Social Optimal Search Intensity in Over-the-Counter Markets*

Shuo Liu†

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Abstract

This paper analyzes OTC market participants' endogeneous search intensity in competitive equilibrium and social optimal cases. We develop a random search-and-match model where agents (market participants) are allowed to choose and adjust their search intensities based on two idiosyncratic states: asset position and liquidity need. We find that: [1] in competitive equilibria with different market parameters, agents can switch between the core and periphery on the trading network. [2] it is the social optimal case that there is no intermediation, in the sense that no agent searches at positive speeds on both the buy and sell sides of the market. In competitive equilibrium, there always exist some agents over-searching and some other agents under-searching. We also discuss related policy implications.

Keywords: endogenous search intensity, social optimal solution, core-periphery network
JEL Classification: G10, G12, G21

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†University of California, Los Angeles, Tsinghua University School of Economics and Management, lsfly0926@ucla.edu

1 Introduction

Over-the-counter (OTC) market played an important role in the 2008 financial crisis. Nearly all of the securities and derivatives involved in the financial turmoil that began with a 2007 breakdown in the U.S. mortgage market were traded in OTC markets. [1] Empirical papers have documented some common stylized facts in OTC markets, one of which is the stable core-periphery trading network. For example, Li and Schürhoff (2014) documents a stable core-periphery structure of dealer network in the U.S. municipal bonds market, through constructing network centrality measures for each dealer; similar market structures are also documented by Hollifield, Neklyudov, and Spatt (2017) for the U.S. securitizations markets, Bech and Atalay (2010) for the federal funds market, and Di Maggio, Kerman, and Song (2017) for the U.S. corporate bond markets.

Existence of the core-periphery trading network can be attributed to agents' heterogeneity in search intensity or meeting technology. In Farboodi, Jarosch, and Shimer (2017b), agents choosing more advanced meeting technologies meet and trade with other agents at a higher frequency, and lie closer to the core of the network. In Neklyudov (2012), meeting technology is interpreted as trading frequency, which is a result of costly investment in customer-relations capital, legal support and the extent of in-house expertise. However, in the current literature, very few papers discuss the formation of core-periphery structure under the assumption that agents endogeneously and ex-post[2] choose their idiosyncratic searth intensities. In this paper, incorporating this assumption into the model allows us to discuss: [**1**] whether agents with similar fundamental characteristics will lie at the same or different position(s) on the trading network, when market parameters change; and [**2**] what are social optimal search intensities at agent level.

In this paper, we construct a search-and-match model with agents endogeneously and ex-post choosing their search intensities, based on their idiosyncratic asset positions and liquidity needs. Using this framework, we discuss the size of the intermediation sector,

[1]Randall Dodd, Markets: Exchange or Over-the-Counter, International Monetary Fund. https://www.imf.org/external/pubs/ft/fandd/basics/markets.htm

[2]Here "ex-post" means whenever agents' idiosyncratic states change, they are allowed tochange their search intensities accordingly. In Farboodi, Jarosch, and Shimer (2017b), agents ex-ante choose their idiosyncratic meeting technologies at the initial time, and each maintains a constant level of meeting technology over time. Therefore, it is the distribution but not the agent-level meeting technologies that are endogeneously and ex-ante determined.

which contains agents searching and trading simultaneously on both the buy and sell sides of the market, in the social optimal solution. In our model, the trading motive between two randomly matched counterparties comes from the difference in their holding positions and private valuations on the target asset. The level of a dealer's private valuation on the asset is proxy for the dealer's liquidity need, and it determines the amount of flow utility the dealer will receive at each time by holding the asset.[3] Therefore, we use the names "private valuation" and "utility type" interchangeably in this paper. Our model is closest to Hugonnier, Lester, and Weill (2018) and Farboodi, Jarosch, and Shimer (2017b). Hugonnier, Lester, and Weill (2018) analyzes the microstructure and trading patterns in OTC market by considering a continuous distribution of trader's private valuations. Also, they maintain the assumption on homogeneous search intensity among all traders. We follow their assumption on $\{0, 1\}$ asset position, that is, agents in our model consecutively switch between the buy side and the sell side of the market, and the trade size between every two counterparties is constantly equal to one. Our innovation is to allow agents to choose idiosyncratic and heterogeneous search intensities. Farboodi, Jarosch, and Shimer (2017b) discusses the formation and welfare consequences of endogenous heterogeneity in trader's search intensity, more from a social planner perspective. In their model setup, each agent's meeting technology is invariant over time after it is chosen at initial time. It is the whole distribution of search intensity that is endogenized. Our model discusses the endogenous heterogeneity more from a competitive equilibrium perspective: agents choose their current search intensity based on their current private valuation and asset position, and we allow agents to adjust their search intensities whenever their private valuations shift up or down and/or their asset positions change through trading with others. In other words, there exists a one-to-one mapping between the two-dimensional states "private valuation and asset position" and search intensity in our model.

Firstly, we show that when market parameters change, agents can "move" on the trading network to switch between the core and periphery positions, even though their private valuations remain unchanged. We focus on competitive and stationary equilibria where the continuous distribution of agents' private valuation is convex and symmetric with respect

[3]Dealers with higher liquidity needs prefer cash to holding the risky asset, and thus their private valuations on the asset are lower than the dealers with lower liquidity needs. This setting of trading motive is consistent with a long and fast growing literature following Duffie, Gârleanu, and Pedersen (2005), which will be discussed more in Section 1.1.

to the intermediate level. Solutions in such equilibria are more interesting and consistent with economic intuition. Specifically, we characterize the shape of search intensity policy function separately for agents on the buy side and agents on the sell side. Then an agent's centrality on the trading network can be measured by her average search intensity across the two sides.[4] To show that agents switch between the core and periphery positions on the network, we do it in the following steps: [1] we show that on the sell side, search intensity is a strictly decreasing function of agent's private valuation. An asset owner[5] with a relatively low private valuation searches at a higher speed than other asset owners with higher private valuations; on the buy side, search intensity is a strictly increasing function of agent's private valuation, which means a nonowner with a relatively high private valuation searches at a higher speed than other nonowners with lower private valuations. All these are consistent with the existence of a competition effect in random search model with multiple agents: a nonowner (owner) with an extremely high (low) valuation has a strong incentive to search faster than others, to correct her mis-aligned asset position through trading with others. [2] Then we characterize the trend of average search intensity among agents of different private valuations, given different market parameters. The parameters determine how costly it is for agents to invest in search intensity and how frequently agents' private valuations change to be higher or lower. We find that, in markets where searching is less costly and/or agents' private valuations change at a lower frequency, the average search intensity is a hump-shaped function of agents' private valuation. The agents with intermediate-level private valuations will *on average* search at a higher speed and lie close to the core of the trading network; however, when searching is more costly and/or agents' private valuations change at a higher frequency, those intermediate-private-valuation agents will on average search and trade at a lower speed than agents with extreme (very high or very low) private valuations. In other words, those intermediate-private-valuation agents switch to the periphery of the trading network, and the extreme-private-valuation agents switch to the core. [3] We calculate different agent-level measures of market liquidity, e.g. gross and intermediation trading volumes, total intermediation profit, intermediation profit per trade, and etc. We find that, agents choosing

[4]This is similar as Neklyudov (2012) who uses trading frequency as a measure of agents' centrality on the trading network. We will show later that agents with higher average search intensity also complete higher total trading volume over the buy and sell sides of the market.

[5]Since we maintain the assumption on $\{0, 1\}$ asset position, an asset owner is an agent who holds one unit of the asset, and a nonowner is an agent who does not hold any position of the asset.

higher average search intensity also trade higher gross volumes, combining the buy and sell sides. This further indicates that average search intensity can be used as proxy for agents' centrality on the trading network. Whenever an agent's average search intensity changes from low to high or in opposite direction, her gross trading volume will accordingly change in the same direction. It is important to note that, for all variables of interest in this paper, we only focus on their relative levels or trends among the agents, instead of their absolute levels.

Secondly, we explicitly solve out the social optimal search intensity functions. The functions imply it is the social optimal case that asset owners with higher-than-intermediate private valuations and asset nonowners with lower-than-intermediate private valuations both search at zero speed. In other words, the size of the intermediation sector is zero, since there does not exist any agent searching at positive speeds simultaneously on both the buy and the sell sides. Moreover, it more benefits the social welfare that agents with extremely mis-aligned asset positions choose higher level of search intensities compared with what they do in competitive equilibrium. Therefore, searching resources are necessarily to be transferred between agents in the social optimal case. This is consistent with the Proposition 2 in Shimer and Smith (2001) that a decentralized competitive equilibrium in a random search environment with multiple agents is not social optimal without taxes. These predictions also depend on the setting of our social welfare objective function. If we define the asset owners with higher-than-intermediate private valuations and the asset nonowners with lower-than-intermediate private valuations as well-aligned agents, then the social level of well-alignment will be the unique part that positively contributes to the social welfare. And the social level of investment cost in search intensities will be the other unique part that negatively contributes to the social welfare. Then it is intuitive that for well-aligned agents, it more benefits the social welfare to make them not search at all to save the investment cost and maintain their current asset positions, until they become mis-aligned ones due to changes in their private valuations; for extremely-misaligned agents, it benefits the social welfare to make them search and trade at higher speeds to reduce the social level of misalignement.[6] Based on these findings, we analytically solve out the explicit-form search intensity functions in the social planner's problem, and they coincide exactly with the numerical solutions. Specifi-

[6]The conclusion that there is no intermediation in the social optimal solution is robust to various forms of search-cost functions. We discuss this in appendix.

cally, in the case of linear search-cost function, we obtain a one-dimention policy measure that a social planner can adopt to achieve the social optimal equilibrium. The social planner only needs to identify a marginal-level private valuation for asset owners, and assign all the asset owners whose private valuations are lower than this marginal level with the maximum level of search intensity; correspondingly, identify another marginal-level private valuation for nonowners, and assign all the nonowners whose private valuations are higher than this marginal level with the same maximum level of search intensity.

Finally, we discuss the appropriate policy response to a specific form of aggregate liquidity shock in our framework. The aggregate liquidity shock changes the distribution of private valuations among the agents. With the occurence of aggregate shock, a certain proportion of agents will have their valuations shifted down by some amount. We consider the policy response as targetting on a certain group of agents to maintain those agents' liquidity needs as their pre-shock levels. In reality, this policy is implemented through directly injecting liquidity into the financial institutions. We find that the policy targetting on agents of higher-than-intermediate private valuations dominates the other ones in perspective of recovering the whole market's liquidity level. Since such group of agents will choose different search intensities in different market environments due to the "switching between the core and periphery on trading network", the appropriate policy response is: in markets with lower frictions, it more benefits the market-level liquidity to firstly inject liquidity into those periphery agents; in markets with higher frictions, it more benefits the market-level liquidity to firstly inject liquidity into those core agents.

Related literature

This paper contributes to the literature initiated by Duffie, Gârleanu, and Pedersen (2005) that applies a search-and-match approach to study asset price and liquidity in OTC markets. Duffie, Gârleanu, and Pedersen (2005) focuses on a general OTC market with investors of only two utility types and an explicit dealer sector. The interdealer market structure is simplified to be a perfect competitive one, which maintains zero inventory position and generates a unique interdealer transaction price. Following this paper, one strand of the literature focuses on fully decentralized market structure, including Duffie, Gârleanu, and Pedersen (2007), Vayanos and Wang (2007), Vayanos and Weill (2008), Weill (2008), Afonso

(2011), Gavazza (2011), Praz (2014), Trejos and Wright (2016), Afonso and Lagos (2015), Atkeson, Eisfeldt, and Weill (2015), Hugonnier, Lester, and Weill (2018), Üslü (2019), Farboodi, Jarosch, and Shimer (2017b), and etc. Another strand of literature focuses on *semi-decentralized* market structure, where transactions between dealers happen in a frictionless centralized market, and transactions between dealers and customers happen in a decentralized market with search frictions, see Weill (2007), Lagos and Rocheteau (2009), Lagos, Rocheteau, and Weill (2011), Feldhütter (2011), Lester, Rocheteau, and Weill (2015), and Pagnotta and Philippon (2018).

Specifically, Duffie, Gârleanu, and Pedersen (2007) studies an OTC market with two types of assets, one paying riskless dividend and the other paying risky dividend. Weill (2008) extends by constructing a multi-asset model, and maintains the assumption that investors' asset positions only take values in $\{0, 1\}$. Afonso and Lagos (2015) focuses on the market for federal funds, and assumes the loan sizes (asset positions) are elements of a countable set. Hugonnier, Lester, and Weill (2018) allows arbitrary continuous distribution of dealers' utility type, and generates intermediation chains and a core-periphery trading network, which is consistent with the empirical findings. They maintain the assumption on exogeneous and homogeneous search intensity among dealers.

There are also papers in this literature considering agents' heterogeneous search intensities: Neklyudov (2012) considers exogeneously heterogeneous search intensity among dealers of two discrete valuation types; Üslü (2019) introduces *ex-ante* heterogeneity in meeting rates into a fully decentralized market model with unrestricted asset holding positions; Farboodi, Jarosch, and Shimer (2017b) consider *ex-ante* choice of the distribution of search intensity at the initial time, after which each agent maintains a fixed level of search intensity over time, even though their private valuations may change afterwards. In this paper, we allow agents to change their search intensities whenever their state variables change.

Our model is different from Shen, Wei, and Yan (2018) who is the first to consider the search intensity decision. They discuss the endogenous entry and exit of investors into an OTC market based on investors' idiosyncratic trading needs and a common search cost, which focuses more on the extensive margin of choosing whether to search or not. Once entering the market, investors will adopt the same level of search intensity. We instead consider agents' intensive margin of choosing how fast to search within the market. Moreover, under the assumption on endogeneous and ex-post search intensity, we explicitly solve out the social optimal one-to-one mapping between agents' idiosyncratic states and search intensities.

This paper also relates to papers which apply network approach to explicitly model the formation of links and bargaining processes between traders in OTC markets, instead of by the search-and-match approach. Related work includes Babus and Kondor (2018), Malamud and Rostek (2017), Barlevy, Alvarez, et al. (2014), Farboodi (2014), Gofman (2011), Chang and Zhang (2018). And there are also some papers (including some papers listed above) combing search and network characteristics, within which Atkeson, Eisfeldt, and Weill (2015) develops a hybrid model to analyze entry and exit equilibrium conditions in the OTC market for credit default swap. With the assumption that traders have homogeneous search intensity, they conclude that banks with both intermediate-level risk exposure (like intermediate-level private valuation in our model) and large size endogenously enter the OTC market and behave as market makers to gain intermediation profit.

There are also papers focusing on alternative, other than search-intensity, mechnisms of endogeneous intermediation, including Farboodi (2014) on bank heterogeneous risk exposure, Neklyudov and Sambalaibat (2015) on dealers' serving clients with different liquidity needs, Wang (2016) on the trade-off between trade competition and inventory efficiency, Farboodi, Jarosch, and Menzio (2017a) on dealers' heterogeneous bargaining power, and Bethune, Sultanum, and Trachter (2018) on private information and heterogeneous screening ability, among others.

Finally, there has been a fast-growing literature on documenting and modeling stylized facts in OTC asset markets. Besides the core-periphery trading network, Li and Schürhoff (2014) also documents the positive correlation between agents' centrality and spreads they earn in municipal bond market, which is also termed as "centrality premium". The correlation of centrality with other statistics such as inventory, trading cost and difference in bargaining power, etc are also discussed. While in securities market for 144a and registered instruments, Hollifield, Neklyudov, and Spatt (2017) documents the negative correlation between agents' centrality and spread, which is termed as "centrality discount". Our model also provides an explanation on the centrality premium and centrality discount, based on agents' switching between the core and periphery on the trading network. Other papers documenting the trading and intermediation structure of OTC markets include but not limited to Di Maggio et al. (2017), Bech and Atalay (2010), Schoar et al. (2014), Siriwardane (2015), and etc.

The rest of the paper is organized as follows: Section 2 lays out the baseline model. Section 3 defines and proves the existence of stationary equilibrium, characterizes equilibria with different model parameters, and discusses the agents' switching between the core and periphery on the trading network. Section 4 analytically solves out the social optimal search intensy functions, and discusses potential policy implications. Section 5 discusses the model's implication for appropriate policy response to a specific form of aggregate liquidity shock. Section 6 concludes.

2 Model

We consider an OTC market for an asset in the form of "consol" which pays one unit of dividend per unit time. This asset is in fixed supply $s = \frac{1}{2}$. There is a continuum of agents $[0, 1]$ who are heterogeneous with respect to utility types $\delta \in [0, 1]$. Agents' utility type measures the amount of utility flow that they receive by holding one unit of the asset.[7] In the cross-section of agents, the cumulative distribution function of utility type is denoted as $F_\delta(\delta)$, and its probability density function is denoted as $f_\delta(\delta)$. Agents' utility types can change to be higher or lower, which happens independently across agents and happens as a Poisson process with intensity α.

Agents randomly search and trade with each other, and consecutively switch between the buy and sell sides. Agents have CARA instantaneous utility as $u(c) = -e^{-\gamma c}$, $\gamma > 0$, their wealth level is denoted as W, and asset position a is restricted as $a \in \{0, 1\}$. Based on their idiosyncratic states (utility type, wealth level and asset position), each agent endogeneously chooses her own search intensity $\lambda \in [0, \bar{\lambda}]$, where $\bar{\lambda}$ is the upper bound of all candidate levels and is same for all agents. To maintain her search intensity as λ, an agent needs to spend a flow cost $C(\lambda) = c_1\lambda^2$, $c_1 > 0$ at each time. In search-and-match process, once two agents meet, their idiosyncratic states determine whether between them there exists a positive trading surplus or not. If there is no positive trading surplus, the two agents depart and continue searching for other trading counterparties; If a trade happens between the two, the transaction price is determined by a Nash bargaining process, and the trade size is restricted to be always one unit asset. We assume agents have the same bargaining power

[7] A higher value of utility type implies an agent has a higher private valuation on holding the asset due to lower liquidity need. An agent with a higher utility type more prefers holding the asset than holding cash.

with each other, denoted as $\theta = \frac{1}{2}$. Specifically, after the trade is completed, the original seller offloads her one unit asset and switches to the buy side; the original buyer takes one unit asset into her inventory and switches to the sell side. Other model parameters are risk free interest rate r and agents' discount rate β.

2.1 Solutions to agent's problem

Let $U(W, \delta, a)$ be the value function of an agent with wealth W, utility type δ and asset position $a \in \{0, 1\}$. Similar as Duffie, Gârleanu, and Pedersen (2007), the agent's problem is:

$$U(W, \delta, a) = \sup_{c,\lambda} E_t \left[-\int_t^{\infty} e^{-\beta(s-t)} e^{-\gamma c_s} ds \middle| W_t = W, \delta_t = \delta, a_t = a \right] \tag{1}$$

$s.t.$

$$dW_t = (rW_t - c_t + a_t\delta_t - C(\lambda_t))dt - P[(W, \delta_t, a_t), (W', \delta'_t, a'_t)]da_t$$

$$\lim_{T\to\infty} e^{-\beta(T-t)} E_t[e^{-r\gamma W_T}] = 0$$

$$C(\lambda_t) = c_1 \lambda_t^2$$

where $P[(W, \delta_t, a_t), (W', \delta'_t, a'_t)]$ is a bilaterally bargained price between two randomly matched counterparties with state variables as (W, δ_t, a_t) and (W', δ'_t, a'_t). da_t is the bilateral trading quantity, and $da_t \in \{-1, 1\}$. In the baseline model, we assume the flow cost $C(\lambda)$ has a quadratic form, and satisfies $C(0) = 0$ and $C'(0) = 0$. We will consider other functional forms of $C(\lambda)$ in the social optimal case.

By guess-and-verify approach similar as Duffie, Gârleanu, and Pedersen (2007), Hugonnier, Lester, and Weill (2018), and Üslü (2019), we simplify the value functions as $V_1(\delta)$ for asset owners and $V_0(\delta)$ for nonowners. The simplied value functions satisfy the following conditions:[8]

[8]It is important to note that, in this paper, we only focus on symmetric equilibria in which all agents adopt the same policy functions $\lambda_1^*(\delta)$ and $\lambda_0^*(\delta)$ to choose their optimal search intensities and form expectations on how search intensities are distributed among all the agents.

For $\forall \delta \in [0,1]$,

$$rV_1(\delta) = \max_{\lambda_1(\delta)} \left\{ \delta - C(\lambda_1(\delta)) + \alpha \int_0^1 (V_1(\delta') - V_1(\delta)) dF_\delta(\delta') \right.$$
$$\left. + \lambda_1(\delta) \int_0^{\bar{\lambda}} \int_0^1 \frac{\lambda'}{\Lambda_0} max\{\triangle V(\delta') - \triangle V(\delta), 0\} \Phi_0(d\delta', d\lambda') \right\} \tag{2}$$

$$rV_0(\delta) = \max_{\lambda_0(\delta)} \left\{ -C(\lambda_0(\delta)) + \alpha \int_0^1 (V_0(\delta') - V_0(\delta)) dF_\delta(\delta') \right.$$
$$\left. + \lambda_0(\delta) \int_0^{\bar{\lambda}} \int_0^1 \frac{\lambda'}{\Lambda_1} max\{\triangle V(\delta) - \triangle V(\delta'), 0\} \Phi_1(d\delta', d\lambda') \right\} \tag{3}$$

where $\triangle V(\delta) = V_1(\delta) - V_0(\delta)$ is the reservation value of an agent with utility type δ, $\Phi_1(\delta', \lambda')$ ($\Phi_0(\delta', \lambda')$) is the cumulative joint measure of utility types and optimal search intensities below (δ', λ') within asset owners (nonowners), $\Lambda_1 = 2\int_0^{\bar{\lambda}} \int_0^1 \lambda' \Phi_1(d\delta', d\lambda')$ is the weighted average search intensity among all asset owners, and $\Lambda_0 = 2\int_0^{\bar{\lambda}} \int_0^1 \lambda' \Phi_0(d\delta', d\lambda')$ is the weighted average search intensity among all nonowners. We use the matching technology discussed by Mortensen (1982), Shimer and Smith (2001) and Üslü (2019). The intensity that an agent with asset position $a \in \{0,1\}$ and search intensity λ_a contacts or is contacted by another agent on the opposite side with asset position $a' \in \{0,1\}$, $a' \neq a$ and search intensity $\lambda'_{a'}$ is $\lambda_a \frac{\lambda'_{a'}}{\Lambda_{a'}} + \lambda'_{a'} \frac{\lambda_a}{\Lambda_a}$. In this paper, we only focus on the case in which asset owners' and nonowners' equilibrium functions are symmetric with respect to the middle-level utility type $\delta = \frac{1}{2}$, therefore we automatically have $\Lambda_a = \Lambda_{a'}$ and the intensity can be simplied as $2\lambda_a \frac{\lambda'_{a'}}{\Lambda_{a'}}$.

Then we obtain the optimal search intensities for asset owners $\lambda_1^*(\delta)$ and nonowners $\lambda_0^*(\delta)$ as follows:

$$\lambda_1^*(\delta) = \frac{\int_0^{\bar{\lambda}} \int_\delta^1 \frac{\lambda'}{\Lambda_0} (\triangle V(\delta') - \triangle V(\delta)) \Phi_0(d\delta', d\lambda')}{2c_1} \tag{4}$$

$$\lambda_0^*(\delta) = \frac{\int_0^{\bar{\lambda}} \int_0^\delta \frac{\lambda'}{\Lambda_1} (\triangle V(\delta) - \triangle V(\delta')) \Phi_1(d\delta', d\lambda')}{2c_1} \tag{5}$$

Proposition 1: *Given the distribution of utility type with symmetric PDF $f_\delta(\delta)$ and the cumulative joint measures $\Phi_0(\delta', \lambda')$ and $\Phi_1(\delta', \lambda')$, in the cross-section of agents: the optimal search intensity chosen by asset owners $\lambda_1^*(\delta)$ is a strictly decreasing function of utility type δ; the optimal search intensity chosen by nonowners $\lambda_0^*(\delta)$ is a strictly increasing function of utility type δ; the reservation value $\triangle V(\delta)$ is a positive-value and strictly increasing function of utility type δ. Proof is in Appendix B.*

By Proposition 1, agents with asset positions more mis-aligned with their utility types will choose to invest in higher level of search intensities, due to their higher gains from searching than agents with asset positions more aligned with utility types. For example, an asset owner with a relatively low utility type has a strong incentive to offload her current inventory as quickly as possible, therefore she invests in a higher level of search intensity than most other agents on the sell side. Once offloading her inventory position to others and if her utility type remains unchanged (i.e. still at a low level), the agent will switch to the buy side and invest in a relatively lower level of search intensity than most other agents on the buy side.

The monotonic properties of the search intensity and reservation value functions $(\lambda_1^*(\delta), \lambda_0^*(\delta), \triangle V(\delta))$ further simplifies the optimal conditions of agent's problem,[9] and combines the two HJB equations (4) and (5) to the single (6) below.

$$r\triangle V(\delta) = \delta + C(\lambda_0^*(\delta)) - C(\lambda_1^*(\delta)) + \alpha \int_0^1 (\triangle V(\delta') - \triangle V(\delta)) dF_\delta(\delta') \tag{6}$$

$$+\lambda_1^*(\delta) \int_\delta^1 \frac{\lambda_0^*(\delta')}{\Lambda_0} (\triangle V(\delta') - \triangle V(\delta)) \phi_0(\delta') d\delta' - \lambda_0^*(\delta) \int_0^\delta \frac{\lambda_1^*(\delta')}{\Lambda_1} (\triangle V(\delta) - \triangle V(\delta')) \phi_1(\delta') d\delta'$$

where

$$\lambda_1^*(\delta) = \frac{\int_\delta^1 \frac{\lambda_0^*(\delta')}{\Lambda_0} (\triangle V(\delta') - \triangle V(\delta)) \phi_0(\delta') d\delta'}{2c_1} \tag{7}$$

[9] Specifically, [1] the increasing reservation value $\triangle V(\delta)$ simplifies the intergral over the full range of utility types, which represents the agent's gains from searching, to be an integral over a subset of utility types. And the subset correlates with the current agent's utility type; [2] since there is a one-to-one mapping between utility type and optimal search intensity on either the buy or sell side, each agent's expectation on the joint distribution of asset positions, utility types and search intensities among all the other agents can thus be simplified as the joint densities of asset position and utility type, which are denoted as $\phi_0(\delta)$ and $\phi_1(\delta)$.

$$\lambda_0^*(\delta) = \frac{\int_0^\delta \frac{\lambda_1^*(\delta')}{\Lambda_1}(\triangle V(\delta) - \triangle V(\delta'))\phi_1(\delta')d\delta'}{2c_1} \tag{8}$$

$$\Lambda_0 = 2\int_0^1 \lambda_0^*(\delta')\phi_0(\delta')d\delta' \tag{9}$$

$$\Lambda_1 = 2\int_0^1 \lambda_1^*(\delta')\phi_1(\delta')d\delta' \tag{10}$$

$$\phi_0(\delta) = \int_0^{\bar{\lambda}} \Phi_0(d\delta, d\lambda') \tag{11}$$

$$\phi_1(\delta) = \int_0^{\bar{\lambda}} \Phi_1(d\delta, d\lambda') \tag{12}$$

2.2 Distribution of agents' idiosyncratic states

Before formally defining the equilibrium, we discuss the equilibrium conditions for the distribution of idiosyncratic states among all agents. Specifically for each utility type δ, we characterize the law of motions for the densities of asset owners $\phi_1(\delta)$ and nonowners $\phi_0(\delta)$. Each agent's utility type changes at Poisson times, and we let $\hat{f}_\delta(\delta)$ denote the PDF of the distribution of the new utility type. The current population distribution satisfies $f_\delta(\delta) = \phi_0(\delta) + \phi_1(\delta)$. For simplicity, we only consider the case $\hat{f}_\delta(\delta) = f_\delta(\delta)$ in equilibrium in next section.[10] The law of motions are as follows:

$$\begin{aligned}\dot{\phi}_1(\delta) = -\alpha\phi_1(\delta) + \frac{\alpha}{2}\hat{f}_\delta(\delta) - 2\phi_1(\delta)\lambda_1^*(\delta)\int_\delta^1 \frac{\lambda_0^*(\delta')}{\Lambda_0}\phi_0(\delta')d\delta' \\ + 2\phi_0(\delta)\lambda_0^*(\delta)\int_0^\delta \frac{\lambda_1^*(\delta')}{\Lambda_1}\phi_1(\delta')d\delta' = 0\end{aligned} \tag{13}$$

[10]In Section 5, we will consider one specific form of aggregate liquidity shock and the refinancing channel defined in Duffie, Gârleanu, and Pedersen (2007). The refinancing channel indicates that $\hat{f}_\delta(\delta) \neq f_\delta(\delta) = \phi_0(\delta) + \phi_1(\delta)$, where the distribution of utility type $f_\delta(\delta)$ can gradually recover to the pre-shock scenario due to the function of $\hat{f}_\delta(\delta)$.

$$\dot{\phi}_0(\delta) = -\alpha\phi_0(\delta) + \frac{\alpha}{2}\hat{f}_\delta(\delta) - 2\phi_0(\delta)\lambda_0^*(\delta)\int_0^\delta \frac{\lambda_1^*(\delta')}{\Lambda_1}\phi_1(\delta')d\delta'$$
$$+ 2\phi_1(\delta)\lambda_1^*(\delta)\int_\delta^1 \frac{\lambda_0^*(\delta')}{\Lambda_0}\phi_0(\delta')d\delta' = 0 \quad (14)$$

In both equation (13) and (14): the first term is the outflow from asset owners (nonowners) due to idiosyncratic liquidity shocks, and the second term is the corresponding inflow; the third term is the outflow due to completed bilateral trades based on random search and match. For example in equation (13), the third term represents asset owners of type δ sell holding positions to their matched nonowners whose utility types are higher than δ; the fourth term is correspondingly the inflow due to completed bilateral trades.

Additionally, $\phi_0(\delta)$ and $\phi_1(\delta)$ at each time also satisfy the following conditions:

$$\phi_0(\delta) + \phi_1(\delta) = f_\delta(\delta) \quad (15)$$

$$\int_0^1 \phi_1(\delta)d\delta = \int_0^1 \phi_0(\delta)d\delta = \frac{1}{2} \quad (16)$$

where (15) is based on the definition of PDF $f_\delta(\delta)$ and the joint densities $\phi_0(\delta)$ and $\phi_1(\delta)$, and (16) is the market clear condition.

3 Stationary equilibrium and core-periphery network

In this section, we firstly define and prove the existance of a stationary equilibrium with an arbitrary population distribution $f_\delta(\delta)$ by Definition 3.1 and Proposition 2, then characterize the shapes of equilibrium functions under different conditions. Finally, we define and characterize the shape of average search intensity function in the cross-section of agents, and discuss its implication for the core-periphery trading network.

Definition 3.1: *A stationary equilibrium contains $\triangle V(\delta)$, $\phi_1(\delta)$, $\phi_0(\delta)$, $\lambda_1^*(\delta)$ and $\lambda_0^*(\delta)$ such that:*

1. *Given $\phi_1(\delta)$, $\phi_0(\delta)$ and $f_\delta(\delta)$, for $\forall\delta \in [0, 1]$:*
   - *$\triangle V(\delta)$, $\lambda_1^*(\delta)$, $\lambda_0^*(\delta)$ solve agent's HJB equation (6) and first order conditions (7)-(8).*

2. *Given* $\triangle V(\delta)$, $\lambda_1^*(\delta)$, $\lambda_0^*(\delta)$, *the endogeneous distributions* $\phi_1(\delta)$, $\phi_0(\delta)$ *satisfy:*

   – $\phi_0(\delta) + \phi_1(\delta) = f_\delta(\delta)$ *for* $\forall \delta \in [0, 1]$.

   – $\dot{\phi}_1(\delta) = \dot{\phi}_0(\delta) = 0$ *in law-of-motion equations* (13)-(14).

3. *Market clears:*

   – $\int_0^1 \phi_1(\delta) d\delta = \frac{1}{2}$.

Proposition 2: *There exists a stationary equilibrium given a uniform distribution of utility type* $f_\delta(\delta) \equiv 1$, $\forall \delta \in [0, 1]$, *for any* $r > 0$, $\alpha > 0$ *and* $c_1 > 0$. *Proof is in Appendix C.*

In Proposition 2, we implicitly assume that the distribution of new utility type $\hat{f}_\delta(\delta)$ by receiving idiosyncratic liquidity shocks is the same as the population distribution $f_\delta(\delta)$ in stationary equilibrium. In Section 5, we relax this assumption to characterize the market dynamics in response to an aggregate liquidity shock.

3.1 Equilibrium with symmetric $f_\delta(\delta)$

In the remaining paper, we consider a specific form of population PDF $f_\delta(\delta)$: symmetric with respect to the middle-level utility type $\delta = \frac{1}{2}$, decreasing in $\delta \in [0, \frac{1}{2}]$, and increasing in $\delta \in [\frac{1}{2}, 1]$. In numerical analysis, we use the uniform distribution as a specific example. The reason we consider a such form of distribution is, when $f_\delta(\delta)$ is convex, the model generates monotonically increasing $\phi_1(\delta)$ and decreasing $\phi_0(\delta)$ in stationary equilibrium. Such an equilibrium is more interesting as it is consistent with the intuition that there is a larger proportion of agents with higher utility types (lower liquidity needs) holding the asset in their inventories. With symmetric PDF $f_\delta(\delta)$, we further define a symmetric stationary equilibrium as follows:

Definition 3.2: *With symmetric PDF* $f_\delta(\delta)$, *in symmetric stationary equilibrium, the density and search intensity functions are required to satisfy:*

$$\phi_0(\delta) = \phi_1(1 - \delta), \qquad \forall \delta \in [0, 1] \tag{17}$$

$$\lambda_0^*(\delta) = \lambda_1^*(1 - \delta), \qquad \forall \delta \in [0, 1] \tag{18}$$

and all equilibrium components $\triangle V(\delta)$, $\phi_1(\delta)$, $\phi_0(\delta)$, $\lambda_1^*(\delta)$ and $\lambda_0^*(\delta)$ also satisfy Definition 3.1.

By Definition 3.2, the agent with the utility type $\delta^* = \frac{1}{2}$ has some special properties: [1] she searches at the same speed on buy and sell sides $\lambda_1^*(\delta^*) = \lambda_0^*(\delta^*)$; [2] her reservation value is exactly equal to the counterfactual frictionless price, i.e., $\triangle V(\delta^*) = \frac{\delta^* + \alpha E_\delta(\triangle V(\delta))}{\alpha + r} = \frac{\delta^*}{r} = p$, where p is the unique market clearing price in the frictionless benchmark. Details about the frictionless benchmarket are in Section A. Intuitively, this agent is indifferent between holding or not holding the asset. By her reservation value, she weights more future utility types than her current asset position. Therefore, her main incentive to enter the market is to provide intermediation services, in the form of purchasing at lower prices and selling at higher prices. We call this agent with $\delta^* = \frac{1}{2}$ as a pure intermediator. Intuitively, the pure intermediator's investment in search intensity should be most sensitive with respect to the market parameters, since she has no inelastic hedging purpose to be either a net buyer or a net seller.

Next we offer the conditions for increasing $\phi_1(\delta)$ and decreasing $\phi_0(\delta)$ in Proposition 3, and we characterize the shapes of the density functions with different model parameters in Proposition 4. Then we are ready to define and characterize the shape of average search intensity $\bar{\lambda}(\delta)$ in Section 3.2. In the expression of $\bar{\lambda}(\delta)$, density functions $\phi_1(\delta)$ and $\phi_0(\delta)$ work as weights imposed on selling-side search intensity $\lambda_1(\delta)$ and buying-side search intensity $\lambda_0(\delta)$.

Proposition 3: *In stationary equilibrium with symmetric (either convex or concave) distribution of utility type $f_\delta(\delta)$, if the following condition is satisfied, we have $\phi_0'(\delta) < 0 < \phi_1'(\delta)$, $\forall \delta \in [0,1]$:*
For $f_\delta(\delta)$, $\nexists\ \delta^* \in [0,1]$ s.t.,

$$\left(\frac{\alpha}{2} + 2\lambda_0^*(\delta) a(\delta^*)\right) f_\delta'(\delta^*) + \frac{1}{c_1}\frac{d\triangle V(\delta^*)}{d\delta}\left(a(\delta^*)^2\phi_0(\delta^*) + b(\delta^*)^2\phi_1(\delta^*)\right) \tag{19}$$

$$+2\lambda_1^*(\delta^*)\lambda_0^*(\delta^*)\phi_1(\delta^*)\phi_0(\delta^*)\left(\frac{1}{\Lambda_0} + \frac{1}{\Lambda_1}\right) = 0$$

where $\lambda_0^*(\delta)$, $\lambda_1^*(\delta)$, Λ_0, Λ_1 and $\triangle V(\delta)$ follow (6)-(10). And notations $a(\delta)$ and $b(\delta)$ follow the Appendix B. Proof is in Appendix D.

The intuition behind condition (19) is: when $f_\delta(\delta)$ is convex, to guarantee $\phi_1'(\delta) > 0$ on $\delta \in [0, \frac{1}{2}]$ (equally $\phi_0'(\delta) < 0$ on $\delta \in [\frac{1}{2}, 1]$), it is necessary that $f_\delta(\delta)$ does not drop too quickly within $\delta \in [0, \frac{1}{2}]$. If $f_\delta(\delta)$ drops too quickly, although there is a larger proportion of agents with high utility types holding the asset, the *absolute* level of density $\phi_1(\delta)$ may still decrease in this range; similarly, when $f_\delta(\delta)$ is concave, to guarantee $\phi_1'(\delta) > 0$ on $\delta \in [\frac{1}{2}, 1]$ (equally $\phi_0'(\delta) < 0$ on $\delta \in [0, \frac{1}{2}]$), it is necessary that $f_\delta(\delta)$ does not to drop too quickly within $\delta \in [\frac{1}{2}, 1]$. Note that $f_\delta(\delta) \equiv 1$, $\forall \delta \in [0, 1]$ automatically satisfies the condition (19) in Proposition 3. In this case, $\phi_0(\frac{1}{2}) = \phi_1(\frac{1}{2}) = \frac{1}{2}$ and $|\phi_0'(\frac{1}{2})| = |\phi_1'(\frac{1}{2})|$. In the remaining part of this paper, we will mainly focus on the case of uniform $f_\delta(\delta)$.

Proposition 4: *In symmetric equilibrium with uniform distribution of utility type $f_\delta(\delta) \equiv 1$, $\forall \delta \in [0, 1]$: if c_1 and/or α increases, given that $\lambda_1^*(0)$ decreases*[11]*, then $\phi_1(\delta)$ ($\phi_0(\delta)$) will increase (decrease) for each $\delta \in [0, \frac{1}{2})$, and will decrease (increase) for each $\delta \in (\frac{1}{2}, 1]$. The magnitude of changes shrink as δ gets closer to the middle-level $\frac{1}{2}$. Proof is in Appendix E.*

The intuition behind Proposition 4 is: when α increases and c_1 remains unchanged , every agent's utility type changes at a higher frequency, then there will be more agents with mis-aligned asset positions.[12] In other words, for each $\delta \in [0, \frac{1}{2})$ ($\delta \in (\frac{1}{2}, 1]$), there will be a larger proportion of asset owners (nonowners); when c_1 increases and α remains unchanged, since it is more costly to search and trade inside the market, there will also be a larger proportion of agents holding mis-aligned asset positions within each $\delta \in [0, 1]$. To summarize, a higher Poisson intensity of idiosyncratic liquidity shocks and/or a more costly investment in searching will raise the market-level mis-alignment of the asset. Figure 1 gives

[11]The reason we need the condition "$\lambda_1^*(0)$ decreases" is that: by increasing the cost coefficient c_1, for asset owner with utility type $\delta = 0$, there will be two counteractive effects that there will be more asset nonowners with utility type higher than zero which potentially increases the benefit from searching but it will also be more expensive for asset owner of type zero to search. "$\lambda_1^*(0)$ decreases" will guarantee that the latter effect dominates. And this will determine the shape of the asset-owner density function since as search is discouraged, there will be more mis-aligned agents in the market; by increasing the parameter α, although the first effect above will encourage the asset owner of type zero to search but there will be more competitors of the same type also with mis-aligned asset positions, which potentially discourages the search at the same time, so it is also reasonable to assume that "$\lambda_1^*(0)$ decreases".

[12]In our model with fixed asset supply $s = \frac{1}{2}$, if the market is frictionless (i.e. Walrasian market), $\phi_0(\delta) = f_\delta(\delta) = 1$ ($\phi_1(\delta) = 0$) for all $\delta \in [0, \frac{1}{2})$ and $\phi_1(\delta) = f_\delta(\delta) = 1$ ($\phi_0(\delta) = 0$) for all $\delta \in [\frac{1}{2}, 1]$, that is, the asset is allocated to the agents who currently value it most. In an OTC market, we call all the asset owners with utility types $\delta \in [0, \frac{1}{2})$ and all the nonowners with utility types $\delta \in [\frac{1}{2}, 1]$ as the ones holding mis-aligned positions. We will discuss the frictionless benchmark in Section 4.

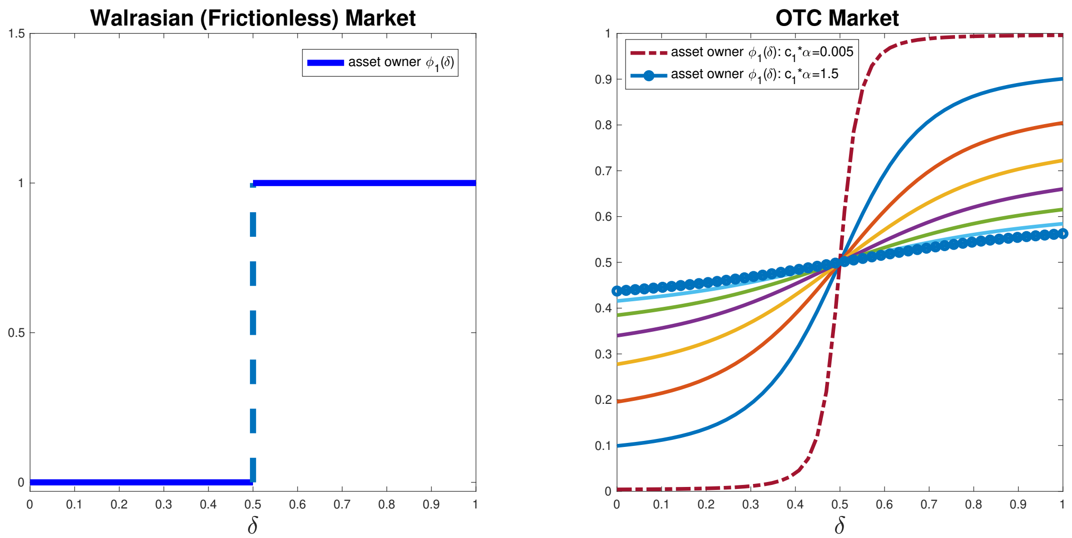

Figure 1: Equilibrium asset-owner density in Walrasian and OTC markets
($\alpha \in [0.005, 0.75]$, $c_1 \in [1, 2]$)

a numerical example of the asset-owner density $\phi_1(\delta)$ under different combinations of α and c_1, in the case of $f_\delta(\delta) \equiv 1$, $\forall \delta \in [0, 1]$. We can see, as the α and/or c_1 shrinks, the shape of asset-owner density will be closer to a frictionless case.

3.2 Average search intensity $\bar{\lambda}(\delta)$

We define the proportions of asset owners and nonowners within each utility type $\delta \in [0, 1]$ as follows:

$$S_0(\delta) = \frac{\phi_0(\delta)}{f_\delta(\delta)} \tag{20}$$

$$S_1(\delta) = \frac{\phi_1(\delta)}{f_\delta(\delta)} \tag{21}$$

Next we define the weighted average search intensity $\bar{\lambda}(\delta)$ as:

$$\bar{\lambda}(\delta) = S_1(\delta)\lambda_1^*(\delta) + S_0(\delta)\lambda_0^*(\delta) = \frac{\phi_1(\delta)}{f_\delta(\delta)}\lambda_1^*(\delta) + \frac{\phi_0(\delta)}{f_\delta(\delta)}\lambda_0^*(\delta) \tag{22}$$

In the case of uniform population distribution $f_\delta(\delta) \equiv 1$, $\forall \delta \in [0, 1]$, the proportions of asset owners and nonowners are exactly densities $\phi_1(\delta)$ and $\phi_0(\delta)$. The shape of $\bar{\lambda}(\delta)$ depends on: [1] each agent's search intensities separately on the buy and the sell side of the market, $\lambda_1^*(\delta)$ and $\lambda_0^*(\delta)$; and [2] the likelihood that each agent lies on the sell side $\phi_1(\delta)$, and the likelihood for the buy side $\phi_0(\delta)$.

Proposition 5: *In symmetric equilibrium with $f_\delta(\delta) \equiv 1$, $\forall \delta \in [0, 1]$, and $c_1 > 0$, $\alpha > 0$, the weighted average search intensity function $\bar{\lambda}(\delta)$ maintains the following properties:*

1. $\bar{\lambda}'(\frac{1}{2}) = 0$, $\bar{\lambda}'(0) < 0$, $\bar{\lambda}'(1) > 0$;
2. *For each $\alpha > 0$ ($c_1 > 0$), $\exists c_1^*(\alpha) > 0$ ($\alpha^*(c_1) > 0$), s.t. if $c_1 > c_1^*(\alpha)$ ($\alpha > \alpha^*(c_1)$):*
   - $\bar{\lambda}'(\delta) < 0 \; \forall \delta \in [0, \frac{1}{2})$;
   - $\bar{\lambda}(0) > \bar{\lambda}(\frac{1}{2})$;
   - $\bar{\lambda}''(\frac{1}{2}) > 0$;
3. *For each $\alpha > 0$ ($c_1 > 0$), $\exists c_1^{**}(\alpha) > 0$ ($\alpha^{**}(c_1) > 0$), s.t. if $c_1 < c_1^{**}(\alpha)$ ($\alpha < \alpha^{**}(c_1)$):*
   - $\exists \hat{\delta} \in (0, \frac{1}{2})$ *s.t.* $\bar{\lambda}'(\hat{\delta}) > 0$;
   - $\bar{\lambda}(0) < \bar{\lambda}(\frac{1}{2})$;
   - $\bar{\lambda}''(\frac{1}{2}) < 0$;

Proof is in Appendix F.

Proposition 5 implies that in different ranges of parameters, the shape of average search intensity $\bar{\lambda}(\delta)$ can be concave (hump-shaped), convex, or in between. Specifically, $\delta = \frac{1}{2}$ is always a stationary point (either local maximum or local minimum point). Given that $\bar{\lambda}'(0) < 0$, in the case of lower c_1, $\delta = \frac{1}{2}$ is a local maximum point. Then by Mean Value Theorem, there must exist a utility type $\delta' \in (0, \frac{1}{2})$ (symmetrically $1 - \delta' \in (\frac{1}{2}, 1)$) which is a local minimum point. With α fixed, when c_1 changes from being small ($< c_1^{**}(\alpha)$) to being large ($> c_1^*(\alpha)$), this local minimum point will shifts from being close to $\delta = 0$ to being close to $\delta = \frac{1}{2}$, on the left part of $\bar{\lambda}(\delta)$, until $\delta = \frac{1}{2}$ becomes the global minimum point. Similar idea works for the case in which c_1 is fixed and α changes from being small ($< \alpha^{**}(c_1)$) to being large ($> \alpha^*(c_1)$).

We can economically understand the shape of average search intensity $\bar{\lambda}(\delta)$ through a composition effect: [1] When search frictions are low, consider an agent with the highest utility type $\delta = 1$. By Proposition 1, when this agent is on the buy side, she is able to search and buy very quickly through investing in an extremely high search intensity. Once she acquires the asset, she switches to the sell side, and also switches to a low-level search intensity, since there are no potential buyers with utility types higher than hers, unless her utility type changes by receiving an idiosyncratic liquidity shock. In other words, when this agent stays on the sell side, the gain from searching is very low. In stationary equilibrium, although this highest-utility-type agent buys very quickly, she spends less time on the buy side (i.e. lower density $\phi_0(\delta)$) and spends more time on the sell side (i.e. higher density $\phi_1(\delta)$). Regarding the densities $\phi_0(\delta)$ and $\phi_1(\delta)$ as weights, her average search intensity is at a low level. Similar result works for an agent with the lowest utility type $\delta = 0$, she sells very quickly and is more likely to be on the buy side, also with a low search intensity. This makes her average search intensity also at a low level. By contrast, the intermediate-utility-type agent ($\delta = \frac{1}{2}$) imposes equal weights on the buy and the sell sides, with relatively high search intensity on both sides when search frictions are low. So considering the average search effort across the two sides, she searches more actively than the other agents. [2] When search frictions are high, even extreme-utility-type agents ($\delta = 0$ and $\delta = 1$) have a weak incentive to invest in high search intensities. This works with the higher frequency of changes in agents' utility types (high α) to make more extreme-utility-type agents have mis-aligned asset positions. Therefore, although the *absolute* level of average search intensity drops, compared with the intermediate-utility-type agents, extreme-type ones still have a stronger "hedging" incentive to search and adjust their mis-aligned positions. And the fact that intermediate-type agents have their average search intensity drop much faster is consistent with their roles as a pure intermediator by Definition 3.2.

To clarify how agents switch between higher and lower search speeds, in the numerical examples and the section of core-periphery trading network, we will mainly focus on the average search intensities and agent-level liquidity measures of the intermediate-utility-type agent ($\delta = \frac{1}{2}$) and the extreme-utility-type agents ($\delta = 0$ and 1).

Numerical examples We give numerical solutions with three sets of parameters c_1 and α in Figure 2-4. The value of c_1 measures how costly it is to invest in searching, and the value of α measures how frequently agents' utility types change. By Proposition 5, as c_1 and α increases, extreme-utility-type agents, who initially invest in search intensities lower than the intermediate-utility-type agents, will switch to search intensities which are higher than the latter agents. Also, by the corresponding graphs of asset-owner and nonowner densities, there will be more agents with mis-aligned asset positions in the market with higher c_1 and α. Note that we are more interested in the trend of average search intensity across different agents. By the numerical examples, we can see the reason why extreme-utility-type agents search at higher speeds than intermediate-utility-type ones is the latter group of agents have their searching speeds drop faster when search frictions are higher.

Figure 2: Equilibrium solutions with $c_1 = 2$, $\alpha = 0.05$

Figure 3: Equilibrium solutions with $c_1 = 5$, $\alpha = 0.25$

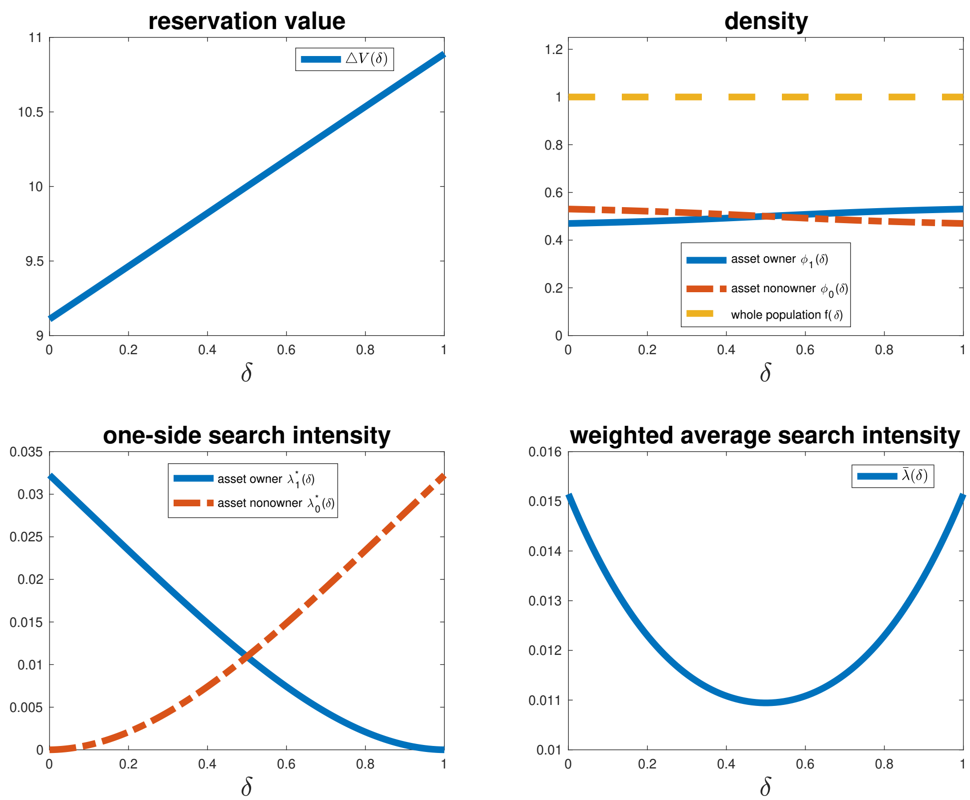

Figure 4: Equilibrium solutions with $c_1 = 10$, $\alpha = 0.5$

3.3 Core-periphery trading network

We follow Neklyudov (2012) to consider the random-search generated trading network. In this section, we show the consistency between agents' average search intensities and gross trading volumes across buy and sell sides. The latter variable is proxy for agents' centrality on the trading network. Additionally, we show the relationship between average search intensity and other agent-level liquidity measures.

Trading volumes We denote agents' gross trading volume across buy and sell sides as $G(\delta)$ in (23), and use it as proxy for agents' centrality. To evaluate the level of intermediation service each agent provides to the whole group, we also define the net volume $N(\delta)$ and intermediation volume $I(\delta)$ in (24)-(25). These three agent-level volume measures are similarly defined and discussed in Neklyudov (2012), Atkeson, Eisfeldt, and Weill (2015) and Üslü (2019).

$$G(\delta) = 2\phi_1(\delta)\lambda_1^*(\delta)\int_\delta^1 \frac{\lambda_0^*(\delta')}{\Lambda_0}\phi_0(\delta')d\delta' + 2\phi_0(\delta)\lambda_0^*(\delta)\int_0^\delta \frac{\lambda_1^*(\delta')}{\Lambda_1}\phi_1(\delta')d\delta' \tag{23}$$

$$N(\delta) = \left|2\phi_1(\delta)\lambda_1^*(\delta)\int_\delta^1 \frac{\lambda_0^*(\delta')}{\Lambda_0}\phi_0(\delta')d\delta' - 2\phi_0(\delta)\lambda_0^*(\delta)\int_0^\delta \frac{\lambda_1^*(\delta')}{\Lambda_1}\phi_1(\delta')d\delta'\right| \tag{24}$$

$$\begin{aligned} I(\delta) &= G(\delta) - N(\delta) \\ &= 4 * min\left\{\phi_1(\delta)\lambda_1^*(\delta)\int_\delta^1 \frac{\lambda_0^*(\delta')}{\Lambda_0}\phi_0(\delta')d\delta', \phi_0(\delta)\lambda_0^*(\delta)\int_0^\delta \frac{\lambda_1^*(\delta')}{\Lambda_1}\phi_1(\delta')d\delta'\right\} \end{aligned} \tag{25}$$

where intermediation volume equals gross volume minus net volume, and it represents the total magnitude of intermediation service that each agent provides to the whole market. Both the gross and intermediation trading volumes are manifestations of agents' ability to reallocate asset positions within the market.

Similar as how the shape of average search intensity $\bar{\lambda}(\delta)$ changes with respect to different market parameters c_1 and α, intermediate-type agents ($\delta = \frac{1}{2}$) switch from the core to the periphery by the measure of gross trading volume in Figure 5, when searching becomes more costly and agents' utility types change at a higher frequency. However by the net and intermediation volumes, their role as a "pure intermediator" remains unchanged in different

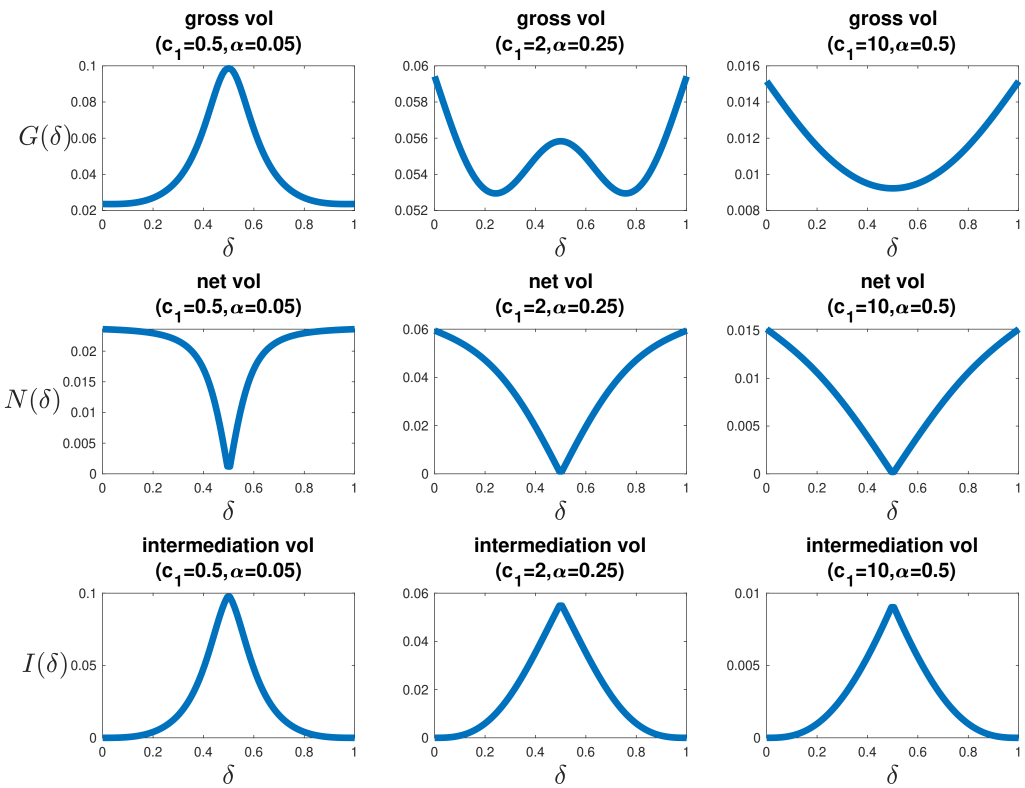

Figure 5: Trading volumes in different markets

market environments.

Centrality profit per trade Our model also sheds some light on the centrality discount and centrality premium separately documented in Li and Schürhoff (2014) and Hollifield, Neklyudov, and Spatt (2017). Specifically, we calculate and characterize the trend of intermediation profit per trade $IP_p(\delta)$, in the cross-section of agents.

$$IP_p(\delta) = \bar{P}_s(\delta) - \bar{P}_b(\delta) \quad \forall \delta \in (0,1) \tag{26}$$

where

$$\bar{P}_s(\delta) = \frac{\int_\delta^1 \frac{\lambda_0^*(\delta')}{\Lambda_0} \frac{\triangle V(\delta) + \triangle V(\delta')}{2} \phi_0(\delta') d\delta'}{\int_\delta^1 \frac{\lambda_0^*(\delta')}{\Lambda_0} \phi_0(\delta') d\delta'} \tag{27}$$

and

$$\bar{P}_b(\delta) = \frac{\int_0^\delta \frac{\lambda_1^*(\delta')}{\Lambda_1} \frac{\triangle V(\delta)+\triangle V(\delta')}{2} \phi_1(\delta') d\delta'}{\int_0^\delta \frac{\lambda_1^*(\delta')}{\Lambda_1} \phi_1(\delta') d\delta'} \tag{28}$$

Intuitively, $\bar{P}_s(\delta)$ and $\bar{P}_b(\delta)$ are agents' average selling price and average buying price[13].

In the current literature, Li and Schürhoff (2014) documents the positive correlation between agents' centrality and bid-ask spread per trade (i.e. centrality premium) in U.S. municipal bond market, and Hollifield, Neklyudov, and Spatt (2017) documents the negative correlation between the two (i.e., centrality discount) in U.S. securitizations market. Similar as Üslü (2019), our model also attributes the different signs of correlation between agents' centrality and intermediation profit per trade to the level of market frictions. The signs of correlation we obtain in different cases are consistent with Üslü (2019). Figure 6 shows the intermediation profit per trade is always minimized at $\delta = \frac{1}{2}$. As a result, centrality premium appears in more frictional and volatile markets, and centrality discount appears in less frictional and volatile markets. By our model, we are able to give a conjesture that the core agents in U.S. securitizations market should have close-to-average liquidity needs, and the core agents in U.S. municipal bond market should on average have either the highest or the lowest liquidity needs among all the agents.

[13]Since agents with $\delta = 0$ ($\delta = 1$) either remain silent or search to sell (buy), they do not provide intermediation service to the whole interagent market. So we ignore these two utility types when discussing intermediation profit per trade.

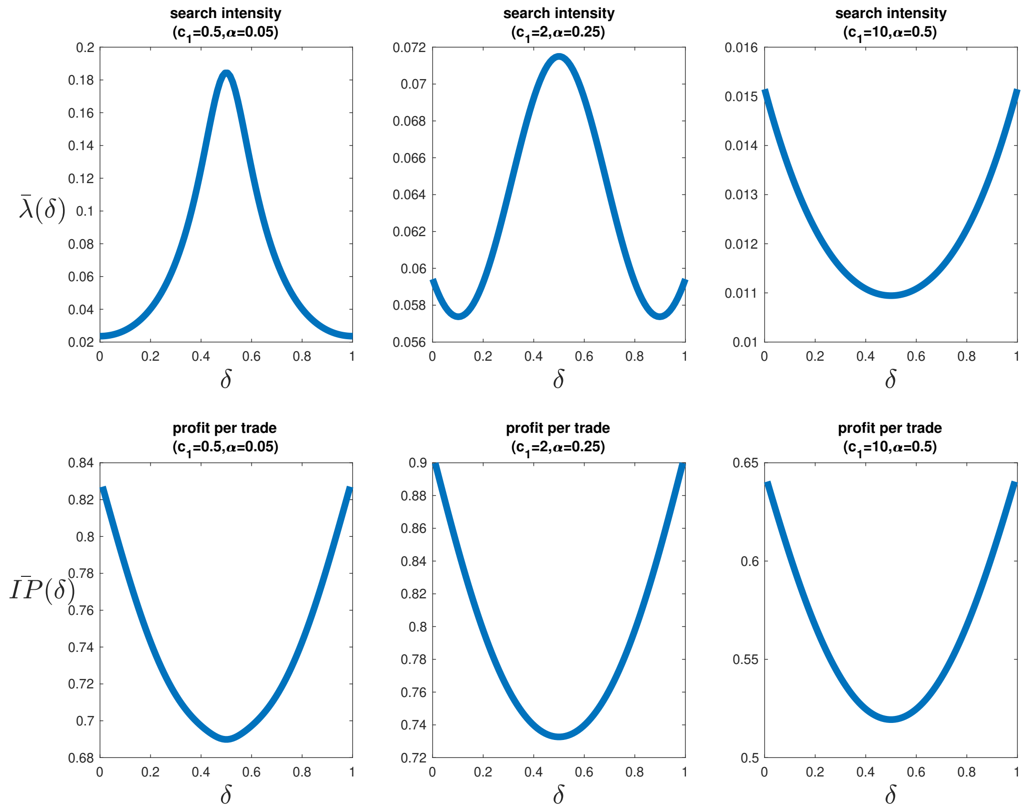

Figure 6: Correlation between centrality and intermediation profit per trade

4 Efficiency analysis

4.1 Social optimal search intensity

We define the social welfare as the difference between the sum of all agents' discounted flow utility and the aggregate search cost. For now, we assume a quadratic form of search cost, and we will relax this assumption in Section 4.2.

$$
\begin{aligned}
W &= \int_0^{+\infty} e^{-rt} \int_0^1 \delta\phi_1(\delta)d\delta dt - \int_0^{+\infty} e^{-rt} \int_0^1 c_1\lambda_1^{*2}(\delta)\phi_1(\delta)d\delta dt \\
&\quad - \int_0^{+\infty} e^{-rt} \int_0^1 c_1\lambda_0^{*2}(\delta)\phi_0(\delta)d\delta dt \\
&= \frac{1}{r}\left(\int_0^1 \delta\phi_1(\delta)d\delta - \int_0^1 c_1\lambda_1^{*2}(\delta)\phi_1(\delta)d\delta - \int_0^1 c_1\lambda_0^{*2}(\delta)\phi_0(\delta)d\delta\right) \qquad (29)
\end{aligned}
$$

We specifically focus on symmetric equilibria, then the last two terms in (29) are equal, and the social welfare is simplified as:

$$
W = \frac{1}{r}\left(\int_0^1 \delta\phi_1(\delta)d\delta - 2\int_0^1 c_1\lambda_1^{*2}(\delta)\phi_1(\delta)d\delta\right) \qquad (30)
$$

Then we discuss that in the symmetric equilibrium with uniform population distribution of utility type $f_\delta(\delta) \equiv 1, \forall\delta \in [0,1]$, what is the social optimal endogeneous search intensities among the agents.[14] We define the following normed linear spaces for the candidate social optimal asset-owner's search intensity $\lambda_1^S(\delta)$ and density $\phi_1^S(\delta)$: $\Lambda_{S1} = \{\lambda_1^S(\delta) : \lambda_1^S(\delta) \in C^1[0,1]; \lambda_1^S(\delta) \geq 0 \text{ and } \lambda_1^{S'}(\delta) \leq 0, \forall\delta \in [0,1]\}$,[15] $\Phi_{S1} = \{\phi_1^S(\delta) : \phi_1^S(\delta) \in C^1[0,1]; 0 \leq$

[14]The original social planner problem has $\lambda_1^{S*}(\delta)$ as its unique control variable. In the final version of social planner's problem, we regard the asset-owner density $\phi_1^S(\delta)$ as the second control function, because there is a one-to-one mapping between $\lambda_1^{S*}(\delta)$ and $\phi_1^S(\delta)$ through the law-of-motion equations of densities in stationary equilibrium.

[15]It is intuitive that the social optimal meeting technology of asset owners $\lambda_1^S(\delta)$ is a decreasing function. Suppose the social optimal function $\lambda_1^{S^*}(\delta)$ has two points $\delta_1 < \delta_2$ with $\lambda_1^{S^*}(\delta_1) < \lambda_1^{S^*}(\delta_2)$, then we can switch the meeting technologies of these two agents without increasing the total investment cost. Then the agent with lower utility type will be assigned with higher meeting technology thus having more oppotunities to sell his asset. Since lower-type asset owners are more likely to be mis-aligned agents, the above switching help improve the alignment of the whole market. Or for simplicity, we can just guess and verify later that

$\phi_1^S(\delta) \leq 1$ and $\phi_1^{S\prime}(\delta) \geq 0, \forall \delta \in [0,1]; \int_0^1 \phi_1^S(\delta)d\delta = \frac{1}{2}\}$, all with the norm $\|f\| = \underset{0\leq\delta\leq 1}{max}|f(\delta)|$. The simplified social planner problem $[SP]$ is:

$$\underset{\lambda_1^S(\delta)\in\Lambda_{S1},\phi_1^S(\delta)\in\Phi_{S1}}{max} \quad W = \int_0^1 (\delta - 2c_1 {\lambda_1^S}^2(\delta))\phi_1^S(\delta)d\delta$$

s.t.

$$\phi_1^S(\delta) = \frac{1}{1 + \frac{\frac{\alpha}{2} + 2\lambda_1^S(\delta)\int_0^{1-\delta}\frac{\lambda_1^S(\delta')}{\Lambda_1}\phi_1^S(\delta')d\delta'}{\frac{\alpha}{2} + 2\lambda_1^S(1-\delta)\int_0^{\delta}\frac{\lambda_1^S(\delta')}{\Lambda_1}\phi_1^S(\delta')d\delta'}} \qquad \forall \delta \in [0,1] \tag{31}$$

and

$$\Lambda_1 = 2\int_0^1 \lambda_1^S(\delta')\phi_1^S(\delta')d\delta'$$

where the constraint (31) is obtained by the law-of-motion equation (13) in stationary equilibrium and also by the symmetry of $\lambda_1^S(\delta)$ and $\lambda_0^S(\delta)$ with respect to $\delta = \frac{1}{2}$, i.e. $\lambda_0^S(\delta) = \lambda_1^S(1-\delta), \forall \delta \in [0,1]$.

Before explicitly solving the social planner problem $[SP]$, we give Proposition 6 which offers a necessary condition on the social optimal solution.

Proposition 6: *If $\lambda_1^{S*}(\delta)$ and $\phi_1^{S*}(\delta)$ solve the social planner problem $[SP]$, then $\lambda_1^{S*}(\delta) \equiv 0$ for $\forall \delta \in [\frac{1}{2}, 1]$. Proof is in Appendix G.*

The intuition behind Proposition 6 is, it is the social optimal case to let only the agents with mis-aligned asset positions to search at positive speeds. For the agents with well-aligned positions, it always benefits the social welfare to make them search at zero speed. To prove Proposition 6, the key idea is to show for any pair of candidate symmetric search intensity functions $\lambda_1^S(\delta)$ and $\lambda_0^S(\delta)$, the objective value W will always increase if we shrink any positive value of $\lambda_1^S(\delta)$ ($\lambda_0^S(\delta)$) in the higher (lower) half range of utility type $[\frac{1}{2}, 1]$ ($[0, \frac{1}{2}]$) and re-assign the shrinked amount of search intensity to a symmetric utility type in the lower (higher) half range.

By Proposition 6, we obtain the explicit-form social optimal search intensity for asset

the social optimal ${\lambda_1^S}^*(\delta)$ is a decreasing function on $[0,1]$.

owners, given a quadratic form of search cost (details are in Appendix I):

$$\lambda_1^{S*}(\delta) = \begin{cases} \frac{-2c_1\alpha^2 + \sqrt{4c_1^2\alpha^4 + 4c_1\alpha^2(\frac{1}{2}-\delta)}}{2c_1\alpha} & \delta \in [0, \frac{1}{2}); \\ 0 & \delta \in [\frac{1}{2}, 1]. \end{cases} \tag{32}$$

We compare the social optimal and competitive equilibrium solutions by a numerical example in Figure 7 and 8. The weighted average search intensity for asset owners (also for nonowners by symmetry) is $\Lambda_1^C = 0.0766$ in competitive equilibrium and $\Lambda_1^S = 0.0376$ in the social optimal solution. This means in the social optimal case, agents on aggregate spend a lower search cost. The social welfare is $W^C = 6.5764$ in competitive equilibrium, which is lower than that in the social optimal solution $W^S = 6.7820$.

Figure 7 compares the search intensity functions between the social optimal and competitive equilibrium solutions. The numerical solution $\lambda_1^{S*}(\delta)$ exactly matches the analytical one in (32). Intuitively, no agents are expected to search at positive speeds on both sides of the market. Also, more searching resources are assigned to agents with mis-aligned asset positions, for example, see $\lambda_1^{S*}(0) > \lambda_1^*(0)$. Figure 8 compares the density functions. We can see the social optimal densities are closer to the Walrasian case in Figure 1. Figure 9 shows that in social optimal solution, the intermediation trading volume is constantly zero across all utility types. This implies there is no intermediation. As a result, the profit per trade is no longer for intermediation services, but trivally equal to the expected revenue (cost) per trade for asset owners (nonowners) with utility types lower (higher) than $\frac{1}{2}$. Moreover, the aggregate of gross trading volumes among all agents is 1.54, compared with 2.33 in the competitive equilibrium solution, also due to the missing of intermediation. All above imply that in competitive equilibrium, agents do not internalize the social externality into their own decisions, and there exist a large amount (in this case, approximately 34%) of inefficient tradings which do not contribute to the well-alignment of the target asset among agents.

Searching resources are necessarily to be transferred between agents in the social optimal case by Proposition 6 and the numerical examples in Figure 7 and 8. In competitive equilibrium, agents with well-aligned asset positions over-search and agents with mis-aligned agents under-search, compared with the social optimal case. This is consistent with the Proposition 2 in Shimer and Smith (2001) that a decentralized competitive equilibrium in a random search environment with multiple agents is not social optimal without taxes.

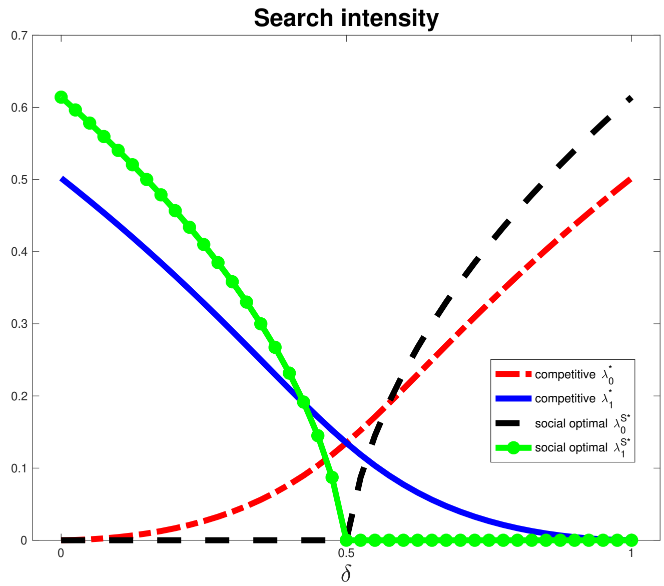

Figure 7: Competitive equilibrium and social optimal search intensities
$(r = 0.05, \alpha = 0.1, c_1 = 1)$

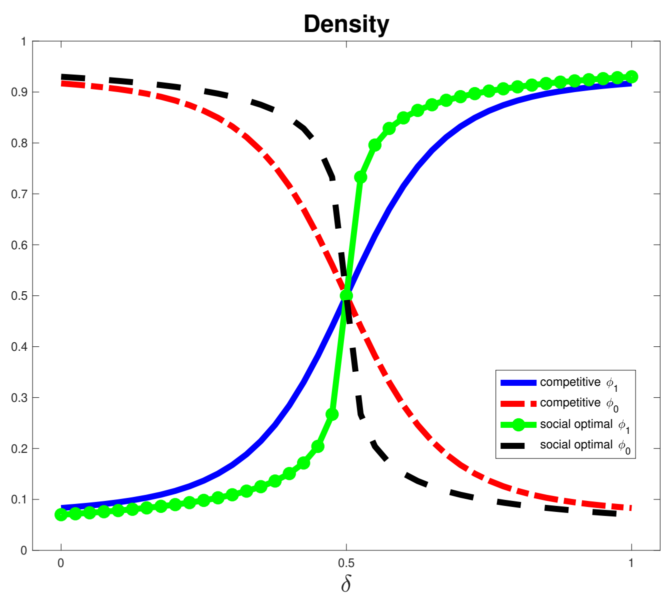

Figure 8: Competitive equilibrium and social optimal densities
$(r = 0.05, \alpha = 0.1, c_1 = 1)$

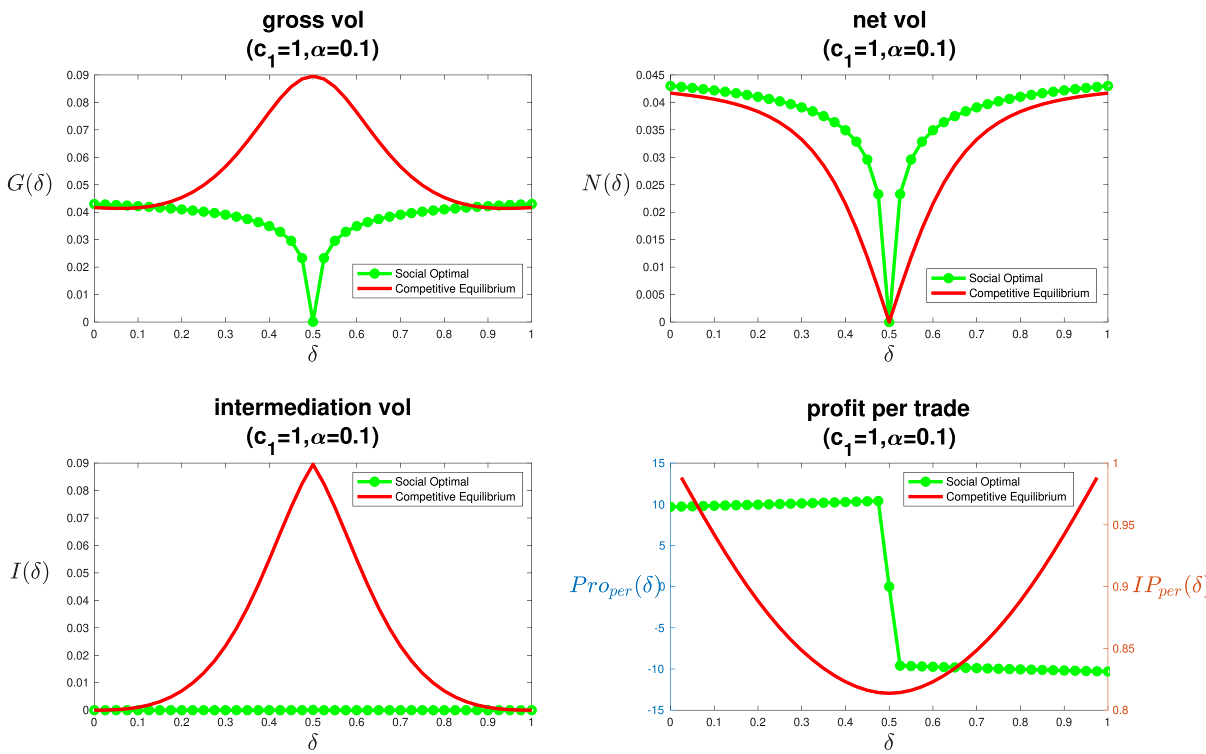

Figure 9: Agent-level liquidity measures

4.2 Robustness check with a general cost function

Section 4.1 essentially concludes that in the social optimal solution, there is no intermediation in the sense that no agents are expected to search at positive speeds on both sides of the market. We generalize this conclusion by considering a general form of cost function in Proposition 7.

Proposition 7: *For any cost function $C(\lambda)$ that satisfies the following condition*[16] [17]

$$C'(\lambda)\begin{cases} \geq 0 & \delta = 0; \\ > 0 & \forall \delta \in (0, \lambda^{ub}]. \end{cases} \tag{33}$$

in symmetric stationary equilibrium, the social optimal search intensity for asset owners satisfies $\lambda_1^{S*}(\delta) \equiv 0$ for $\forall \delta \in [\frac{1}{2}, 1]$. Proof is in Appendix H.

Condition (33) applies for most cost functions including the quadratic form $C(\lambda) = c_1\lambda^2$, linear form $C(\lambda) = c_1\lambda$, concave form $C(\lambda) = c_1\lambda^p$, $p \in (0, 1)$, and etc, in all cases $c_1 > 0$.

The key this result applies is our assumption that agents are allowed to switch to new search intensities in response to idiosyncratic liquidity shocks. Compared with Farboodi, Jarosch, and Shimer (2017b), in their paper, each agent' search intensity remains constant over time. This is equivalent that the adjustment cost is infinity. Therefore, there is no one-to-one mapping between utility type and search intensity. Also, the main trading incentive in their paper comes from the difference in search intensities between every two matched agents, and the agent with the more advanced search intensity will play the role of intermediator. In our paper, agents are allowed to adjust their search intensities without any adjustment cost. This makes it possible for an agent to search at the maximum speed on one side of the market, and freely switches to zero speed on the other side of the market, and vice versa.

Next we show how to obtain the explicit expression of $\lambda_1^{S*}(\delta)$, for a general cost function $C(\lambda)$ satisfying the condition in Proposition 7. By substituting $\lambda_1^{S*}(\delta) \equiv 0, \forall \delta \in [\frac{1}{2}, 1]$ into the equilibrium constraint to obtain the expression of $\phi_1^S(\delta)$, we obtain the reduced-form

[16] In condition (33), $C'(0) \geq 0$ includes the case that $C'(0) = +\infty$

[17] λ^{ub} is the upper bound of candidate search intensities for either asset owners or nonowners. If there is no upper bound, it is equal that $\lambda^{ub} = +\infty$.

social planner problem $[RP]$:

$$\max_{\lambda_1^S(\delta)\in\Lambda_{S1}} \quad W^*(\lambda_1^S(\delta)) = \int_0^{\frac{1}{2}} (\delta - 2C(\lambda_1^S(\delta)))\frac{1}{1+\frac{\frac{\alpha}{2}+\lambda_1^S(\delta)}{\frac{\alpha}{2}}}d\delta + \int_{\frac{1}{2}}^1 \delta\frac{1}{1+\frac{\frac{\alpha}{2}}{\frac{\alpha}{2}+\lambda_1^S(1-\delta)}}d\delta$$
$$= \int_0^{\frac{1}{2}} \left(\frac{-\alpha C(\lambda_1^S(\delta)) + \frac{\alpha}{2} + (1-\delta)\lambda_1^S(\delta)}{\alpha+\lambda_1^S(\delta)}\right)d\delta$$
$$= \int_0^{\frac{1}{2}} f^C(\lambda_1^S(\delta),\delta)d\delta$$

s.t.

$$K_{\frac{1}{2}} = \int_0^{\frac{1}{2}} \frac{\frac{\alpha}{2}\lambda_1^S(\delta)}{\alpha+\lambda_1^S(\delta)}d\delta \leq \Lambda \tag{34}$$

where in (34), Λ is the restricted maximum search intensity of the whole market.
By Hamiltonian approach,

$$L(\delta,\lambda_1^S(\delta)) = H(\delta,K_\delta) + \mu(\Lambda - K_{\frac{1}{2}})$$
$$= \frac{-\alpha C(\lambda_1^S(\delta)) + \frac{\alpha}{2} + (1-\delta)\lambda_1^S(\delta)}{\alpha+\lambda_1^S(\delta)} + m_\delta\frac{\frac{\alpha}{2}\lambda_1^S(\delta)}{\alpha+\lambda_1^S(\delta)} + \mu(\Lambda - K_{\frac{1}{2}}) \tag{35}$$

where

$$K_\delta = \int_0^{\delta} \frac{\frac{\alpha}{2}\lambda_1^S(\delta')}{\alpha+\lambda_1^S(\delta')}d\delta' \quad \text{and} \quad \dot{K}_\delta = \frac{\frac{\alpha}{2}\lambda_1^S(\delta)}{\alpha+\lambda_1^S(\delta)} \tag{36}$$

The necessary conditions for $\lambda_1^{S*}(\delta) : [0,\frac{1}{2}) \to R^+$ to be the optimal solution are:

$$\dot{m}_\delta = -\frac{\partial H(\delta,K_\delta)}{\partial K_\delta} = 0 \tag{37}$$

$$\bar{m} = m_{\frac{1}{2}} = \begin{cases} 0 & \text{if } \mu = 0; \\ -\frac{\partial W^*(\lambda_1^S(\delta))}{\partial K_{\frac{1}{2}}} & \text{if } \mu > 0. \end{cases} \tag{38}$$

For simplicity, we consider the case $\Lambda = \lambda^{ub}$, i.e., the constraint in (34) is never binding, then $\mu = 0$ and $\bar{m} = 0$ by (38). Additionally, we have:

$$\frac{\partial L(\delta, \lambda_1^S(\delta))}{\partial \lambda_1^S(\delta)} = \frac{\partial}{\partial \lambda_1^S(\delta)} \left(\frac{-\alpha C(\lambda_1^S(\delta)) + \frac{\alpha}{2} + (1-\delta)\lambda_1^S(\delta)}{\alpha + \lambda_1^S(\delta)} \right)$$
$$= \frac{\frac{\alpha}{2} - \alpha\delta + \alpha C(\lambda_1^S(\delta)) - \alpha(\alpha + \lambda_1^S(\delta))C'(\lambda_1^S(\delta))}{(\alpha + \lambda_1^S(\delta))^2} \tag{39}$$

and

$$\frac{\partial^2 L(\delta, \lambda_1^S(\delta))}{\partial \lambda_1^{S^2}(\delta)} = \frac{-\alpha C''(\lambda_1^S(\delta))(\alpha + \lambda_1^S(\delta))^2}{(\alpha + \lambda_1^S(\delta))^3}$$
$$- \frac{(\alpha - 2\alpha\delta + 2\alpha C(\lambda_1^S(\delta)) - 2\alpha(\alpha + \lambda_1^S(\delta))C'(\lambda_1^S(\delta)))}{(\alpha + \lambda_1^S(\delta))^3} \tag{40}$$

For each $\delta \in [0, \frac{1}{2})$, the social optimal search intensity $\lambda_1^{S^*}(\delta)$ takes different ranges of values, given the following different conditions:

1. $\frac{\partial L(\delta,\lambda_1^S(\delta))}{\partial \lambda_1^S(\delta)}|_{\lambda_1^S(\delta)=\lambda_1^{S^*}(\delta)} = 0$ and $\frac{\partial^2 L(\delta,\lambda_1^S(\delta))}{\partial \lambda_1^{S^2}(\delta)}|_{\lambda_1^S(\delta)=\lambda_1^{S^*}(\delta)} \leq 0$: then $0 < \lambda_1^{S^*}(\delta) < \lambda^{ub}$;

2. $\frac{\partial L(\delta,\lambda_1^S(\delta))}{\partial \lambda_1^S(\delta)} > 0$ for $\forall \lambda_1(\delta) \in [0, \lambda^{ub}]$: then $\lambda_1^{S^*}(\delta) = \lambda^{ub}$;

3. $\frac{\partial L(\delta,\lambda_1^S(\delta))}{\partial \lambda_1^S(\delta)} < 0$ for $\forall \lambda_1(\delta) \in [0, \lambda^{ub}]$: then $\lambda_1^{S^*}(\delta) = 0$;

4. For every $\delta \in [0, \frac{1}{2})$, $\nexists \lambda_1^S(\delta) \in [0, \lambda^{ub}]$ s.t. $\frac{\partial L(\delta,\lambda_1^S(\delta))}{\partial \lambda_1^S(\delta)} = 0$ and $\frac{\partial^2 L(\delta,\lambda_1^S(\delta))}{\partial \lambda_1^{S^2}(\delta)} \leq 0$, and $W(\lambda_1^S(\delta) \equiv 0) > W(\lambda_1^S(\delta) \equiv \lambda^{ub})$: then $\lambda_1^{S^*}(\delta) \equiv 0$, $\forall \delta \in [0, \frac{1}{2})$;

5. For every $\delta \in [0, \frac{1}{2})$, $\nexists \lambda_1^S(\delta) \in [0, \lambda^{ub}]$ s.t. $\frac{\partial L(\delta,\lambda_1^S(\delta))}{\partial \lambda_1^S(\delta)} = 0$ and $\frac{\partial^2 L(\delta,\lambda_1^S(\delta))}{\partial \lambda_1^{S^2}(\delta)} \leq 0$, and $W(\lambda_1^S(\delta) \equiv 0) < W(\lambda_1^S(\delta) \equiv \lambda^{ub})$: then $\lambda_1^{S^*}(\delta) \equiv \lambda^{ub}$, $\forall \delta \in [0, \frac{1}{2})$.

In Appendix I, we solve the explicit-form $\lambda_1^{S*}(\delta)$ separately for each of the following cost functions: quadratic forms $C(\lambda) = c_1\lambda^2$ and $C(\lambda) = c_2\lambda^2 + c_3\lambda$, $(c_2 < 0, c_3 > 0)$, linear form $C(\lambda) = c_1\lambda$, and concave form $C(\lambda) = c_1\lambda^p$, $p \in (0, 1)$, where in all cases $c_1 > 0$.

4.3 One-dimentional policy measure with linear search cost

In this section, we specifically focus on the social optimal solution with a linear cost function $C(\lambda) = c_1\lambda$. The solution offers a one-dimention policy measure that a social planner can adopt to achieve the social optimal equilibrium. The social planner only needs to identify a marginal-level utility type for asset owners, and assign all the asset owners with utility types lower than this marginal level with the maximum level of search intensity; correspondingly, identify another marginal-level utility type for nonowners, and assign all the nonowners with utility types higher than this marginal level with the same maximum level of search intensity.

Proposition 8: *In social planner problem with $C(\lambda) = c_1\lambda$ $(c_1 > 0)$, the social optimal search intensity $\lambda_1^{S*}(\delta)$ satisfies:*

1. *If $c_1\alpha < \frac{1}{2}$,*

$$\lambda_1^{S*}(\delta) = \begin{cases} \lambda^{ub} & \text{if } \delta \leq \delta_1^*; \\ 0 & \text{if } \delta > \delta_1^*. \end{cases} \tag{41}$$

where $\delta_1^* = \frac{1}{2} - c_1\alpha$ is the marginal-level utility type among asset owners;

2. *If $c_1\alpha \geq \frac{1}{2}$,*

$$\lambda_1^{S*}(\delta) \equiv 0 \quad \forall \delta \in [0, 1] \tag{42}$$

Details are in Appendix I.

Figure 10 gives a numerical example to compare the search intensities between the social optimal and competitive equilibrium solutions, given a linear search cost. We can see that there is no intermediation in the social optimal solution, and there is even a small group of agents with utility types around $\delta = \frac{1}{2}$ being fully silent on both sides of the market.

5 Aggregate liquidity shock

We consider an aggregate liquidity shock with a similar form as Duffie, Gârleanu, and Pedersen (2007). In their paper, upon receiving the aggregate shock, a randomly chosen proportion of agents suffer a sudden drop in their utility types. In our model, since the population distribution of utility type $f_\delta(\delta)$ has continuous support $[0, 1]$, we consider the aggregate liquidity shock in a new form such that, for each agent whose utility type is within $\delta \in [\frac{1}{2}, 1]$, upon the occurence of aggregate shock, each agent's utility type shifts to $\delta - \frac{1}{2}$ with a probability π. The shifting occurs independently among all the agents in $\delta \in [\frac{1}{2}, 1]$, which allows us to

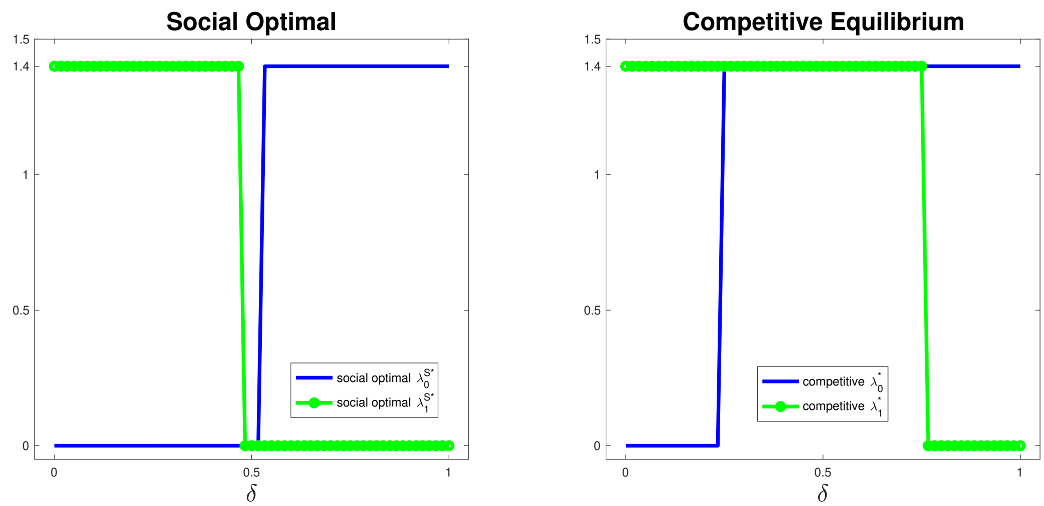

Figure 10: Social optimal and competitive equilibrium search intensities for linear cost ($r = 0.05, c_1 = 0.05, \alpha = 0.35, \lambda^{ub} = 1.4$, and social welfares are $W^S = 6.8953$, $W^C = 6.2031$.)

apply the Law of Large Numbers. Figure 11 shows the changes in population distribution and density functions upon the aggregate shock.

We maintain the self-refinancing channel in Duffie, Gârleanu, and Pedersen (2007) such that the distribution of each agent's new utility type in response to idiosyncratic liquidity shocks is assumed to be always uniform on $[0, 1]$, and the population distribution of utility type can recover to the pre-shock scenario through this channel. Also, we assume the aggregate liquidity shock arrives at Poisson times and have a permanent effect on asset price through driving agents' expectations.

Assuming t is the length of time after the most recent aggregate liquidity shock, we obtain the new HJB equations for agents who are indirectly affected $\delta \in [0, \frac{1}{2})$ and agents who are directly affected $\delta \in [\frac{1}{2}, 1]$:

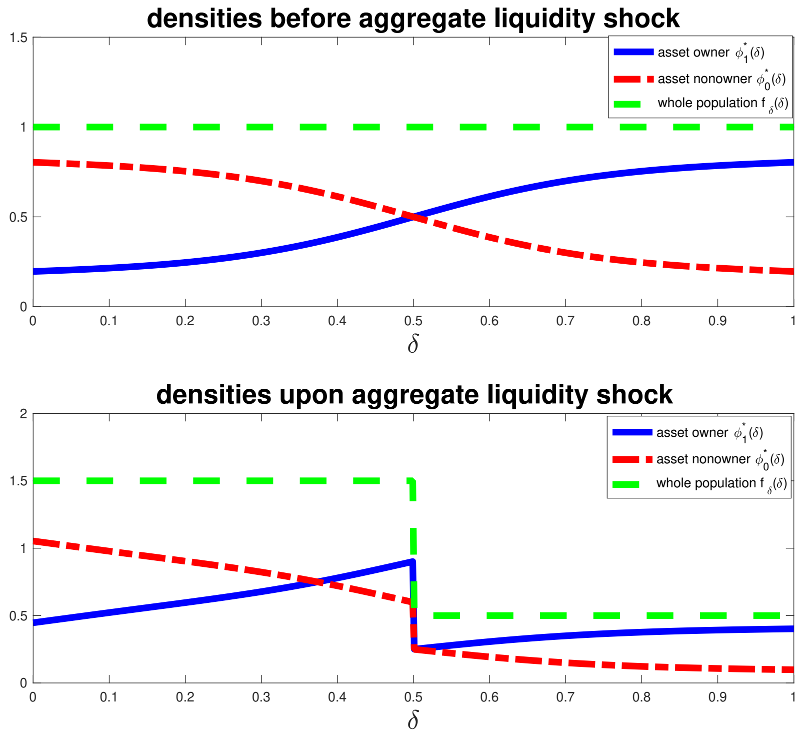

Figure 11: Densities before and after the aggregate liquidity shock (with $\pi = 0.5$)

For $\forall \delta \in [0, \frac{1}{2}]$,

$$\begin{aligned}\triangle\dot{V}(\delta,t) &= r\triangle V(\delta,t) - \delta + c_1\lambda_1^{*2}(\delta,t) - c_1\lambda_0^{*2}(\delta,t) - \alpha\int_0^1(\triangle V(\delta',t) - \triangle V(\delta,t))d\delta' \\ &- \lambda_1^*(\delta,t)\int_0^1 \frac{\lambda_0^*(\delta',t)}{\Lambda_{0,t}}(\triangle V(\delta',t) - \triangle V(\delta,t))\phi_0(\delta',t)d\delta' \\ &+ \lambda_0^*(\delta,t)\int_0^\delta \frac{\lambda_1^*(\delta',t)}{\Lambda_{1,t}}(\triangle V(\delta,t) - \triangle V(\delta',t))\phi_1(\delta',t)d\delta' \\ &- \eta(\triangle V(\delta,0) - \triangle V(\delta,t)) \qquad (43)\end{aligned}$$

For $\forall \delta \in [\frac{1}{2}, 1]$,

$$\begin{aligned}
\triangle \dot{V}(\delta,t) &= r\triangle V(\delta,t) - \delta + c_1 {\lambda_1^*}^2(\delta,t) - c_1 {\lambda_0^*}^2(\delta,t) - \alpha \int_0^1 (\triangle V(\delta',t) - \triangle V(\delta,t))d\delta' \\
&- \lambda_1^*(\delta,t) \int_0^1 \frac{\lambda_0^*(\delta',t)}{\Lambda_{0,t}} (\triangle V(\delta',t) - \triangle V(\delta,t))\phi_0(\delta',t)d\delta' \\
&+ \lambda_0^*(\delta,t) \int_0^\delta \frac{\lambda_1^*(\delta',t)}{\Lambda_{1,t}} (\triangle V(\delta,t) - \triangle V(\delta',t))\phi_1(\delta',t)d\delta' \\
&- \eta \left[\pi(\triangle V(\delta - 0.5, 0) - \triangle V(\delta,t)) + (1-\pi)(\triangle V(\delta,0) - \triangle V(\delta,t))\right] \qquad (44)
\end{aligned}$$

where η is the expected Poisson intensity of future aggregate liquidity shock.

The law-of-motion equations and market clear condition for densities $\phi_1(\delta,t)$ and $\phi_0(\delta,t)$ after the aggregate liquidity shock are as follows:

For $\forall t > 0$ and $\forall \delta \in [0, 1]$,

$$\dot{\phi_1}(\delta,t) = -\alpha\phi_1(\delta,t) + \frac{\alpha}{2}\hat{f}_\delta(\delta) - 2\phi_1(\delta,t)\lambda_1^*(\delta,t) \int_\delta^1 \frac{\lambda_0^*(\delta',t)}{\Lambda_{0,t}}\phi_0(\delta',t)d\delta' \qquad (45)$$

$$+2\phi_0(\delta,t)\lambda_0^*(\delta,t) \int_0^\delta \frac{\lambda_1^*(\delta',t)}{\Lambda_{1,t}}\phi_1(\delta',t)d\delta'$$

$$\dot{\phi_0}(\delta,t) = -\alpha\phi_0(\delta,t) + \frac{\alpha}{2}\hat{f}_\delta(\delta) + 2\phi_1(\delta,t)\lambda_1^*(\delta,t) \int_\delta^1 \frac{\lambda_0^*(\delta',t)}{\Lambda_{0,t}}\phi_0(\delta',t)d\delta' \qquad (46)$$

$$-2\phi_0(\delta,t)\lambda_0^*(\delta,t) \int_0^\delta \frac{\lambda_1^*(\delta',t)}{\Lambda_{1,t}}\phi_1(\delta',t)d\delta'$$

where $\hat{f}_\delta(\delta) \equiv 1 \ \forall \delta \in [0, 1]$ and

$$\phi_0(\delta,t) + \phi_1(\delta,t) = f_\delta(\delta,t) \qquad (47)$$

$$\int_0^1 \phi_0(\delta,t)d\delta = \int_0^1 \phi_1(\delta,t)d\delta = \frac{1}{2} \qquad (48)$$

The aggregate shock generates a permanent effect on transaction prices and market liquidity. In Figure 12 and 13, we show the trends of market average selling and buying prices, and other agent-level liquidity measures, before and after the aggregate liquidity shock. [1] Due to agents' expectations on the occurence of future aggregate shocks, the new equilibrium prices are permanently lower than the levels in the normal situation. [2] Right after the occurence of aggregate liquidity shock, there are immediate increases in market-level gross and intermediation volumes. The reason is right after the start of crisis, agents with their utility types shifted down have a strong incentive to sell their asset positions to higher-type agents, and the latter group also expects higher gains from searching. These motivate all the agents to re-allocate the asset positions between themselves, which will immediately raise the gross and intermediation volumes in the market. As time goes by, both volumes will go down since agents achieve better-aligned asset positions. Similarly due to the expectation on future aggregate liquidity shock, agents are less incentivized to search and trade with each other in the new stationary equilibrium. [3] Right after the occurence of aggregate liquidity shock, there is an immediate decrease in the cross-agent average level of intermediation profit (bid-ask spread) per trade. The decrease is due to the decline in the cross-agent average utility type. Although it usually measures the market-level transaction cost, the decrease does not necessarily imply an improvement in market liquidity, since agents may trade-off between lower average transaction cost and a higher trading delay, which is beyond the discussion of this paper.

Policy targeting on different groups of agents We consider a specific form of policy response such that the regulatory authority directly injects liquidity into a targeted group of agents, to make the affected agents' liquidity needs (i.e. utility types) recover to their pre-shock levels. We consider two policy choices: one targets on all the agents with $\delta \in [\frac{1}{2}, \frac{3}{4}]$ (Policy#1), and the other targets on all the agents with $\delta \in [\frac{3}{4}, 1]$ (Policy#2). Then we characterize the trends of market-level liquidity measures under these two policies, to decide which policy is more effective in maintaining the market-level liquidity. By Figure 14, Policy#2 uniformly dominates Policy#1 and the "no policy response" choice, in terms of different liquidity measures and across different levels of market friction. Intuitively, agents with higher utility types contribute more to maintaining the market liquidity. The key implication of our model is, such higher-type agents become core agents in a market where searching is more costly and/or agents' utility type change at a higher frequency; and they become periphery agents in the opposite market environment. Then we conclude that the

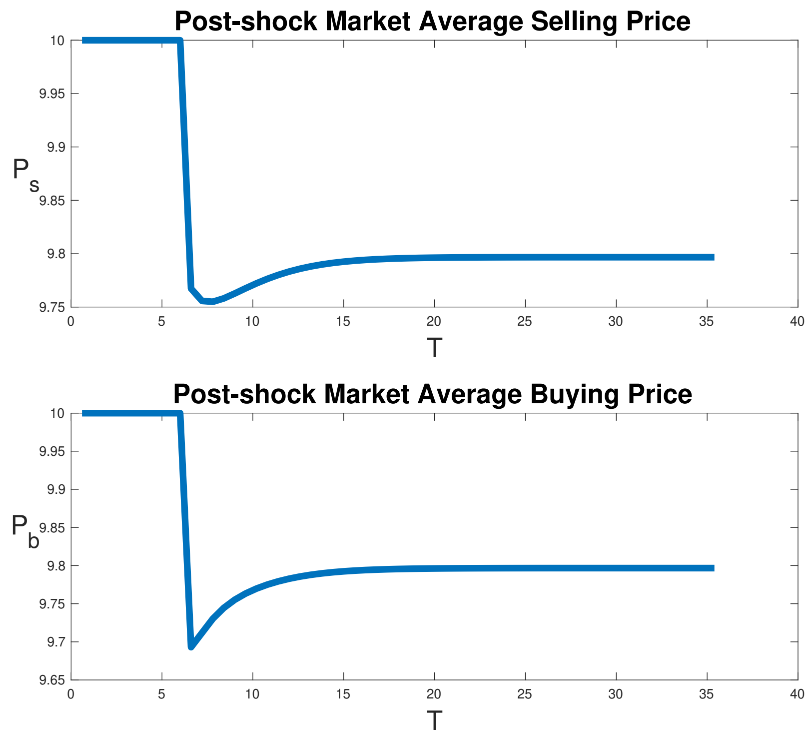

Figure 12: Market average prices with $\alpha = 0.25$, $c_1 = 1$ and $\eta = 0.1$

core agents may not always be the most important ones that are given the priority to receive injection of liquidity after the aggregate liquidity shock. To better maintain the market-level liquidity, policy makers need to firstly identify the market parameters.

6 Conclusion

This paper develops a random search-and-match model where agents are allowed to endogeneously choose and adjust their search intensities based on idiosyncratic states. This model can generate the core-periphery trading network. We characterize the competitive equilibria with different market parameters, and discuss its implication for the formation of core-periphery trading network. Then we explicitly solve out the social optimal solution under a general form of search cost function. Our main conclusions include: [1] agents can switch between the core and periphery on the trading network. In markets where searching

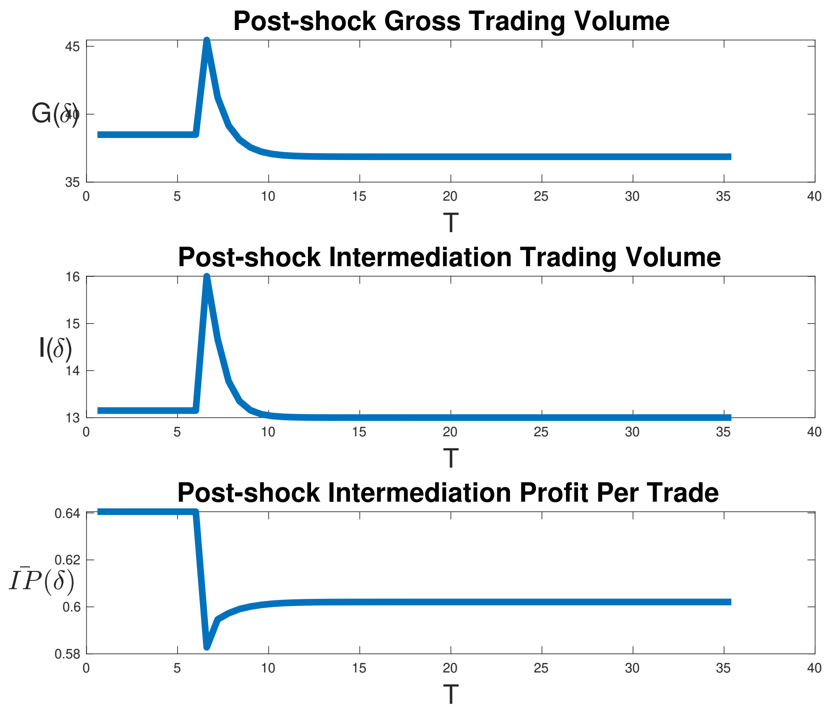

Figure 13: Measures of market liquidity with $\alpha = 0.25$, $c_1 = 1$ and $\eta = 0.1$

is more costly and agents' utility types change at a higher frequency, periphery agents are more important in perspective of maintaining the market-level liquidity; while in the opposite market environments, core agents are more important. [2] in the social optimal case, there is no intermediation, in the sense that no agent searches at positive speeds on both sides of the market.

In this paper, we implicitly assume that there is perfect information in the market since every agent has a rational expectation on the population distribution of utility types. As a result, the main searching motive in our model is either to gain intermediation profits or to hedge mis-aligned asset positions. Since the two most significant characteristics of OTC markets are search frictions and imperfect information, in future research, we could incorporate private information into the model and consider an alternative searching motive to learn from trading. (e.g. to learn the quality of target asset or the trading counterparty's private valuation.)

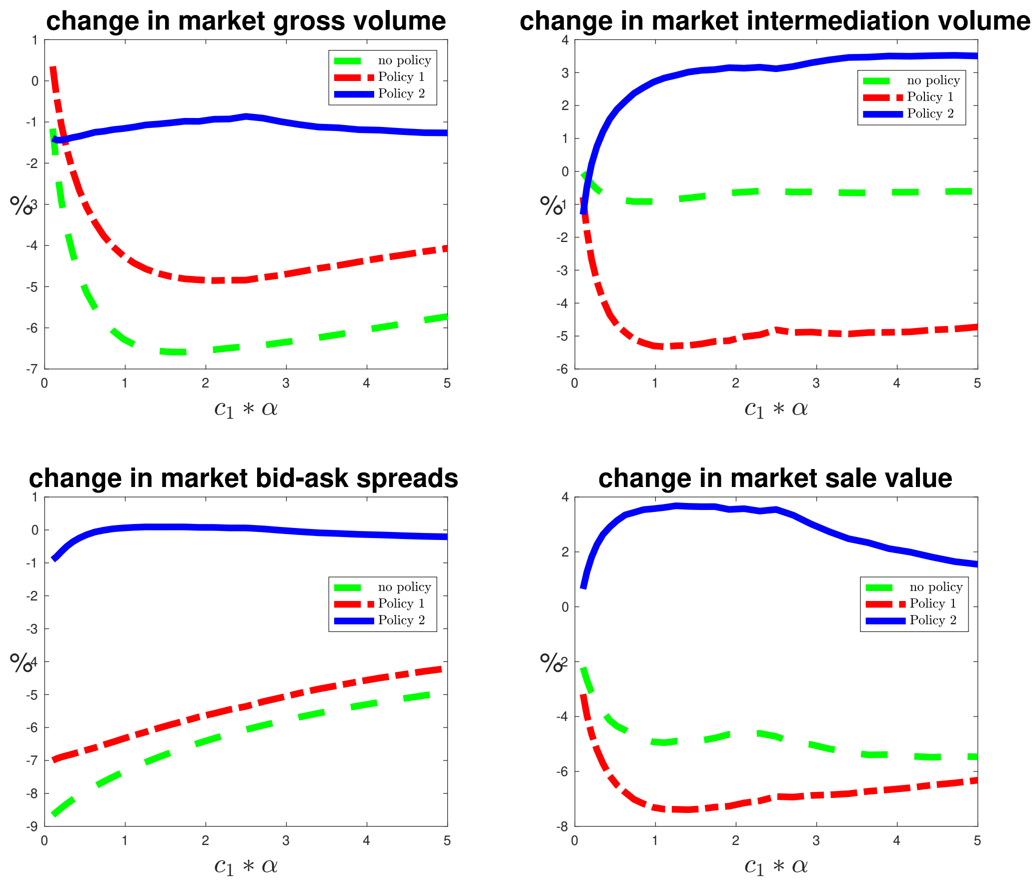

Figure 14: Cumulative changes before achieving a new stationary equilibrium (cumulative changes are expressed as a percentage of the old equilibrium)

Appendices

A The frictionless benchmark

To understand the effects of OTC market search frictions on market efficiency, we firstly characterize the frictionless benchmark, i.e., the Walrasian market. In this market, upon receiving idiosyncratic liquidity shocks, every agent can adjust her asset position immediately to accommodate her new utility type at the unique price p.

Assume one agent's current asset position is $a \in \{0,1\}$, and her new asset position after adjustment is $a' \in \{0,1\}$. The frictionless version of reservation value function $V_a^f(\delta)$ satisfies the following HJB equation:

$$rV_a^f(\delta) = \delta * a + \alpha \int_0^1 \underset{a'}{max} \left[V_{a'}^f(\delta') - V_a^f(\delta) - p(a' - a) \right] dF_\delta(\delta') \tag{49}$$

By first order condition of a', we have:

$$a' = \begin{cases} 1 & \text{if } \triangle V^f(\delta') > p; \\ 1 \text{ or } 0 & \text{if } \triangle V^f(\delta') = p; \\ 0 & \text{if } \triangle V^f(\delta') < p. \end{cases}$$

Since $\triangle V^f(\delta)$ is strictly increasing in δ by (49), $\exists! \delta^* \in [0,1]$ s.t. $\triangle V^f(\delta^*) = p$. Also, since p is the market clearing price, as in Hugonnier, Lester, and Weill (2018), we have the following market clear condition:

$$\delta^* = inf\{\delta \in [0,1] : 1 - F_\delta(\delta) \leq \frac{1}{2}\} \tag{50}$$

which means for all the agents with utility types $\delta > \delta^*$, they will hold the asset, and for all the agents with utility types $\delta < \delta^*$, they will not hold the asset. In other words, $\phi_1^f(\delta) = f(\delta)$, $\forall \delta \in [\delta^*, 1]$ and $\phi_0^f(\delta) = f(\delta)$, $\forall \delta \in [0, \delta^*]$. We call δ^* as the marginal-level utility type in Walrasian market. In OTC markets, we regard all asset owners with utility types in $[0, \delta^*]$ and all nonowners with utility types in $[\delta^*, 1]$ as the agents with mis-aligned asset positions. In the case of fixed asset supply $s = \frac{1}{2}$, we have $\delta^* = \frac{1}{2}$.

The HJB equation (49) then reduces to:

$$r\triangle V^f(\delta) = \delta + \alpha(p - \triangle V^f(\delta)) \tag{51}$$

Then for $\delta^* = \frac{1}{2}$:

$$\triangle V^f(\delta^*) = \frac{\delta^*}{r} = \frac{1}{2r} = p \tag{52}$$

We let $a^f(\delta)$ denote the new asset position for $\delta \in [0,1]$ after adjustment. At each time point, the expected instantaneous total trading volume in Walrasian market is:

$$\begin{aligned}
TV^f &= \alpha \int_0^1 \int_0^1 |a^f(\delta') - a^f(\delta)| dF_\delta(\delta') dF_\delta(\delta) \\
&= 2\alpha \int_0^{\delta^*} \int_{\delta^*}^1 |a^f(\delta') - a^f(\delta)| dF_\delta(\delta') dF_\delta(\delta) \\
&= 2\alpha(1 - F_\delta(\delta^*))F_\delta(\delta^*) \\
&= 2\alpha s(1-s) \\
&= \frac{\alpha}{2}
\end{aligned}$$

Intuitively in such a frictionless market, all the tradings happen due to idiosyncratic liquidity shocks, and all of the tradings are completed between the agents and the Walrasian auctioner. If we sum up all agents' continuation utilities, we can obtain the social welfare with $f_\delta(\delta) \equiv 1, \forall \delta \in [0,1]$:

$$\begin{aligned}
W^f &= \int_0^{+\infty} e^{-rt} \left(\int_0^1 \delta \phi_1^f(\delta) d\delta \right) dt \\
&= \frac{1}{r} \int_{\delta^*}^1 \delta f_\delta(\delta) d\delta \\
&= \frac{E[\delta; \delta > \delta^*]}{r} \\
&= \frac{3}{8r}
\end{aligned}$$

B Proposition 1

We use guess-and-verify approach to prove the monotonicity of reservation value function $\triangle V(\delta)$. Suppose $\triangle V(\delta)$ is strictly increasing, then with (2) minus (3) we obtain (6):

$$r\triangle V(\delta) = \delta + C(\lambda_0^*(\delta)) - C(\lambda_1^*(\delta)) + \alpha \int_0^1 (\triangle V(\delta') - \triangle V(\delta)) dF_\delta(\delta')$$

$$+\lambda_1^*(\delta) \int_\delta^1 \frac{\lambda_0^*(\delta')}{\Lambda_0} (\triangle V(\delta') - \triangle V(\delta)) \phi_0(\delta') d\delta' - \lambda_0^*(\delta) \int_0^\delta \frac{\lambda_1^*(\delta')}{\Lambda_1} (\triangle V(\delta) - \triangle V(\delta')) \phi_1(\delta') d\delta'$$

and also (7)-(10). By (6)-(10), we obtain that for $\forall \delta \in [0,1]$

$$(\triangle V(\delta))^2 \left(\frac{a(\delta)^2 - b(\delta)^2}{4c_1} \right) + \triangle V(\delta) \left(r + \alpha + \frac{B(\delta)b(\delta) - A(\delta)a(\delta)}{2c_1} \right)$$
$$- \delta - \alpha E[\triangle V] - \frac{B(\delta)^2 - A(\delta)^2}{4c_1} = 0 \tag{53}$$

where

$$A(\delta) = \int_0^\delta \frac{\lambda_1^*(\delta')}{\Lambda_1} \triangle V(\delta') \phi_1(\delta') d\delta' \tag{54}$$
$$B(\delta) = \int_\delta^1 \frac{\lambda_0^*(\delta')}{\Lambda_0} \triangle V(\delta') \phi_0(\delta') d\delta' \tag{55}$$
$$a(\delta) = \int_0^\delta \frac{\lambda_1^*(\delta')}{\Lambda_1} \phi_1(\delta') d\delta' \tag{56}$$
$$b(\delta) = \int_\delta^1 \frac{\lambda_0^*(\delta')}{\Lambda_0} \phi_0(\delta') d\delta' \tag{57}$$
$$E[\triangle V] = \int_0^1 \triangle V(\delta') f_\delta(\delta') d\delta' \tag{58}$$

We denote the LHS of equation (53) as F, by Implicit Function Theorem, we verify that

$$\frac{d\triangle V(\delta)}{d\delta} = -\frac{\partial F/\partial \delta}{\partial F/\partial \triangle V(\delta)} = \frac{1}{r + \alpha + \lambda_1^*(\delta) b(\delta) + \lambda_0^*(\delta) a(\delta)} > 0, \quad \forall \delta \in [0,1] \tag{59}$$

By first order conditions (7)(8), it is trival that

$$\lambda_1^*(1) = \lambda_0^*(0) = 0 \tag{60}$$

and by plugging the first order conditions into the HJB equations, the optimal search intensity functions $\lambda_1^*(\delta)$ and $\lambda_0^*(\delta)$ satisfy

$$(\alpha + r)V_1(\delta) = \delta + c_1\lambda_1^{*2}(\delta) + \alpha E[V_1(\delta)] \tag{61}$$

$$(\alpha + r)V_0(\delta) = c_1\lambda_0^{*2}(\delta) + \alpha E[V_0(\delta)] \tag{62}$$

where the expectation $E[\cdot]$ is by a symmetric PDF $f_\delta(\delta)$,
(61)-(62) $\Longrightarrow$

$$(\alpha + r)\triangle V(\delta) = \delta + c_1\lambda_1^{*2}(\delta) - c_1\lambda_0^{*2}(\delta) + \alpha E[\triangle V(\delta)] \tag{63}$$

apply $E[\cdot]$ on both sides $\Longrightarrow$

$$(\alpha + r)E[\triangle V(\delta)] = E[\delta] + c_1\left(\int_0^1 \lambda_1^{*2}(\delta)f_\delta(\delta)d\delta - \int_0^1 \lambda_0^{*2}(\delta)f_\delta(\delta)d\delta\right) + \alpha E[\triangle V(\delta)] \tag{64}$$

Later we will prove that if $f_\delta(\delta)$ is symmetric with respect to $\delta = \frac{1}{2}$, then the equilibrium optimal search intensity functions $\lambda_1^*(\delta)$ and $\lambda_0^*(\delta)$ will be symmetric to each other with respect to $\delta = \frac{1}{2}$, i.e. $\lambda_0^*(\delta) = \lambda_1^*(1-\delta)$ for $\forall\delta \in [0,1]$. Here we just take this conclusion as given and we get:

$$\int_0^1 \lambda_1^{*2}(\delta)f_\delta(\delta)d\delta = \int_0^1 \lambda_0^{*2}(\delta)f_\delta(\delta)d\delta \tag{65}$$

Together with (64), we obtain:

$$E[\triangle V(\delta)] = \frac{E(\delta)}{r} > 0 \tag{66}$$

Then by (60)(63),

$$(\alpha + r)\triangle V(0) = c_1\lambda_1^{*2}(0) + \alpha E[\triangle V(\delta)] > 0 \tag{67}$$

Together with $\frac{d\triangle V(\delta)}{d\delta} > 0, \quad \forall\delta \in [0,1]$, we obtain

$$\triangle V(\delta) > 0, \quad \forall\delta \in [0,1] \tag{68}$$

By (7)(8), we obtain:

$$\frac{d\lambda_1^*(\delta)}{d\delta} = \frac{-\frac{d\triangle V(\delta)}{d\delta}\int_\delta^1 \frac{\lambda_0^*(\delta')}{\Lambda_0}\phi_0(\delta')d\delta'}{2c_1} < 0 \quad \forall\delta \in [0,1] \tag{69}$$

$$\frac{d\lambda_0^*(\delta)}{d\delta} = \frac{\frac{d\triangle V(\delta)}{d\delta}\int_0^\delta \frac{\lambda_1^*(\delta')}{\Lambda_1}\phi_1(\delta')d\delta'}{2c_1} > 0 \quad \forall\delta \in [0,1] \tag{70}$$

C Proposition 2

Based on properties of the competitive equilibrium components $\triangle V(\delta)$, $\lambda_1^*(\delta)$ and $\phi_1(\delta)$, with $f_\delta(\delta) \equiv 1$, we define the following normed linear spaces: $\triangle V_S = \{\triangle V(\delta) : \triangle V(\delta) \in C^1[0,1]; \triangle V(\delta) \geq 0 \text{ and } \triangle V'(\delta) > 0, \forall\delta \in [0,1]; E(\triangle V(\delta)) = \int_0^1 \triangle V(\delta)d\delta = \frac{1}{2r}\}$, $\Lambda_{S1} = \{\lambda_1^*(\delta) : \lambda_1^*(\delta) \in C^1[0,1]; \lambda_1^*(\delta) \geq 0 \text{ and } \lambda_1^{*'}(\delta) < 0, \forall\delta \in [0,1]\}$, $\Phi_{S1} = \{\phi_1(\delta) : \phi_1(\delta) \in C^1[0,1]; 0 \leq \phi_1(\delta) \leq 1 \text{ and } \phi_1'(\delta) > 0, \forall\delta \in [0,1]; \int_0^1 \phi_1(\delta)d\delta = \frac{1}{2}\}$, all with the norm $\|f\| = \underset{0\leq\delta\leq1}{max}|f(\delta)|$.

The vector of stationary equilibrium components $[\triangle V(\delta) \quad \lambda_1^*(\delta) \quad \lambda_0^*(\delta) \quad \phi_1(\delta) \quad \phi_0(\delta)]^T$, by symmetry between $\lambda_1^*(\delta)$ and $\lambda_0^*(\delta)$ and symmetry between $\phi_1(\delta)$ and $\phi_0(\delta)$, is a fixed point of the following transformation $T : \triangle V_S \times \Lambda_{S1} \times \Phi_{S1} \longrightarrow \triangle V_S \times \Lambda_{S1} \times \Phi_{S1}$:[18]

$$T\begin{bmatrix} \triangle V(\delta) \\ \lambda_1^*(\delta) \\ \phi_1(\delta) \end{bmatrix} = \begin{bmatrix} T_1(\triangle V(\delta)) \\ T_2(\lambda_1^*(\delta)) \\ T_3(\phi_1(\delta)) \end{bmatrix} \tag{71}$$

where

$$\begin{aligned} T_1(\triangle V(\delta)) =& \frac{\delta + c_1\lambda_0^{*2}(\delta) - c_1\lambda_1^{*2}(\delta) + \alpha\int_0^1 \triangle V(\delta')dF_\delta(\delta')}{r + \alpha + \lambda_1^*(\delta)\int_\delta^1 \frac{\lambda_0^*(\delta')}{\Lambda_0}\phi_0(\delta')d\delta' + \lambda_0^*(\delta)\int_0^\delta \frac{\lambda_1^*(\delta')}{\Lambda_1}\phi_1(\delta')d\delta'} \\ &+ \frac{\lambda_1^*(\delta)\int_\delta^1 \frac{\lambda_0^*(\delta')}{\Lambda_0}\triangle V(\delta')\phi_0(\delta')d\delta' + \lambda_0^*(\delta)\int_0^\delta \frac{\lambda_1^*(\delta')}{\Lambda_1}\triangle V(\delta')\phi_1(\delta')d\delta'}{r + \alpha + \lambda_1^*(\delta)\int_\delta^1 \frac{\lambda_0^*(\delta')}{\Lambda_0}\phi_0(\delta')d\delta' + \lambda_0^*(\delta)\int_0^\delta \frac{\lambda_1^*(\delta')}{\Lambda_1}\phi_1(\delta')d\delta'} \\ =& \frac{1}{\alpha + r}\left(\delta + c_1\lambda_1^{*2}(\delta) - c_1\lambda_1^{*2}(1-\delta) + \frac{\alpha}{2r}\right) \end{aligned}$$

[18] For each "component mapping" $(T_1 - T_3)$, we assume all the other equilibrium components are given and may/may not be the corresponding "fixed points".

$$T_2(\lambda_1^*(\delta)) = \frac{1}{2c_1}\int_\delta^1 \frac{\lambda_0^*(\delta')}{\Lambda_0}\triangle V(\delta')\phi_0(\delta')d\delta'$$
$$= \frac{1}{2c_1}\int_0^{1-\delta} \frac{\lambda_1^*(\delta')}{\Lambda_1}(\triangle V(1-\delta') - \triangle V(\delta))\phi_1(\delta')d\delta'$$
$$\text{s.t. } \Lambda_1 = 2\int_0^1 \lambda_1^*(\delta')\phi_1(\delta')d\delta'$$

$$T_3(\phi_1(\delta)) =$$
$$\frac{\alpha\int_0^1 \lambda_1^*(\delta')\phi_1(\delta')d\delta' + 2\lambda_1^*(1-\delta)\int_0^\delta \lambda_1^*(\delta')\phi_1(\delta')d\delta'}{2\alpha\int_0^1 \lambda_1^*(\delta')\phi_1(\delta')d\delta' + 2\lambda_1^*(1-\delta)\int_0^\delta \lambda_1^*(\delta')\phi_1(\delta')d\delta' + 2\lambda_1^*(\delta)\int_0^{1-\delta} \lambda_1^*(\delta')\phi_1(\delta')d\delta'}$$

We use $(\overline{\triangle V}(\delta), \overline{\lambda_1^*}(\delta), \overline{\phi_1}(\delta))$ to denote the fixed points of $\triangle V(\delta)$, $\lambda_1^*(\delta)$ and $\phi_1(\delta)$. Then we verify that, given any two of the three fixed points, the third one always exists.

(1) Given the fixed point $\overline{\lambda_1^*}(\delta)$, by transformation T_1,
$\overline{\triangle V}(\delta) = \frac{1}{\alpha+r}\left(\delta + c_1\overline{\lambda_1^*}^2(\delta) - c_1\overline{\lambda_1^*}^2(1-\delta) + \frac{\alpha}{2r}\right)$ is a fixed point of $\triangle V(\delta)$.

(2) Given the fixed points $\overline{\triangle V}(\delta)$ and $\overline{\phi_1}(\delta)$, plug them into transformation T_2 which is trivally continuous, we can prove this T_2 works on normed linear space Λ_{S1} which is nonempty (trivally), convex and compact.

Convexity

For $\forall \lambda_1^{1*}(\delta), \lambda_1^{2*}(\delta) \in \Lambda_{S1}$ and $\forall \lambda \in (0,1)$, define the new function $\hat{\lambda}(\delta) = \lambda * \lambda_1^{1*}(\delta) + (1-\lambda) * \lambda_1^{2*}(\delta)$, it is trival that $\hat{\lambda}(\delta) \in C^1[0,1]$, $\hat{\lambda}(\delta) \geq 0$ and $\hat{\lambda}'(\delta) = \lambda * \lambda_1^{1*\prime}(\delta) + (1-\lambda) * \lambda_1^{2*\prime}(\delta) < 0$. So $\hat{\lambda}(\delta) \in \Lambda_{S1}$ for $\forall \lambda \in (0,1)$.

Boundedness

For $\forall \lambda_1^*(\delta) \in \Lambda_{S1}$,

$$\begin{aligned} T_2(\lambda_1^*(\delta)) &= \frac{1}{2c_1}\int_0^{1-\delta} \frac{\lambda_1^*(\delta')}{\Lambda_1}(\overline{\triangle V}(1-\delta') - \overline{\triangle V}(\delta))\overline{\phi_1}(\delta')d\delta' \\ &\leq (\overline{\triangle V}(1) - \overline{\triangle V}(0))\frac{1}{2c_1}\int_0^{1-\delta} \frac{\lambda_1^*(\delta')}{\Lambda_1}\phi_1(\delta')d\delta' \\ &\leq (\overline{\triangle V}(1) - \overline{\triangle V}(0))\frac{1}{2c_1}\int_0^{1} \frac{\lambda_1^*(\delta')}{\Lambda_1}\phi_1(\delta')d\delta' \\ &= \frac{(\overline{\triangle V}(1) - \overline{\triangle V}(0))}{2c_1} \end{aligned}$$

By Proposition 1, $\forall \triangle V(\delta) \in \triangle V_S$ is strictly increasing on $[0, 1]$. We have,

$$0 < \triangle V(1) - \triangle V(0) = \frac{1 - 2c_1\lambda_1^{*2}(0)}{\alpha + r} < \frac{1}{\alpha + r} \tag{72}$$

then

$$T_2(\lambda_1^*(\delta)) < \frac{1}{2c_1(\alpha + r)} \tag{73}$$

Equicontinuity

Firstly we need to prove the boundedness of $\frac{d\triangle V(\delta)}{d\delta}$ for $\forall \triangle V(\delta) \in \triangle V_S$.

$$\frac{d\triangle V(\delta)}{d\delta} = \frac{1}{r + \alpha + \lambda_1^*(\delta)b(\delta) + \lambda_0^*(\delta)a(\delta)} \tag{74}$$

where $0 \leq a(\delta) = \int_0^\delta \frac{\lambda_1^*(\delta')}{\Lambda_1}\phi_1(\delta')d\delta' \leq \frac{1}{2}$ and $0 \leq b(\delta) = \int_\delta^1 \frac{\lambda_0^*(\delta')}{\Lambda_0}\phi_0(\delta')d\delta' \leq \frac{1}{2}$.
Also by (72), $0 < \lambda_1^*(0) = \underset{\delta\in[0,1]}{max}\lambda_1^*(\delta) < \frac{1}{\sqrt{2c_1}}$, then we get

$$\frac{1}{r + \alpha + \frac{1}{\sqrt{2c_1}}} < \frac{d\triangle V(\delta)}{d\delta} < \frac{1}{r + \alpha} = B_{dV} \tag{75}$$

Then for $\forall \lambda_1^*(\delta) \in \Lambda_{S1}$ and $\forall \delta \in [0, 1]$: given $\forall \epsilon > 0$, we can always choose small enough $\hat{\triangle} = \frac{2c_1\epsilon}{B_{dV}} > 0$, such that, by (7)-(10),

$$|\lambda_1^*(\delta + \hat{\triangle}) - \lambda_1^*(\delta)| = \left|\frac{-\hat{\triangle}}{2c_1}\frac{d\overline{\triangle V}(\delta)}{d\delta}\int_\delta^1 \frac{\lambda_0^*(\delta')}{\Lambda_0}\phi_0(\delta')d\delta' + o(\hat{\triangle})\right| \leq 2 * \frac{1}{2c_1}\frac{2c_1\epsilon}{B_{dV}}B_{dV}\frac{1}{2} = \epsilon.$$

Since $\hat{\triangle}$ does not relate to specific value of δ, then any sequence of functions in normed linear space Λ_{S1} is uniform equicontinuous on $[0, 1]$. Based on **boundedness** and **equicontinuity** above, and refer to Arzelä-Ascoli theorem in Arzelà (1895), we prove that the continuous transformation T_2, given the fixed points $\overline{\triangle V}(\delta)$ and $\overline{\phi_1}(\delta)$, maps Λ_{S1} to Λ_{S1}. And the normed linear space Λ_{S1} is nonempty, convex and compact. By Schauder's fixed point theorem, given the fixed points $\overline{\triangle V}(\delta)$ and $\overline{\phi_1}(\delta)$, there exists a fixed point $\overline{\lambda_1^*}(\delta)$ of $\lambda_1^*(\delta)$.

(3) Given the fixed points $\overline{\lambda_1^*}(\delta)$ and $\overline{\triangle V}(\delta)$, plug $\overline{\lambda_1^*}(\delta)$ into transformation T_3 which is trivally continuous, we can prove this T_3 works on the normed linear space Φ_{S1} which is nonempty, convex and compact.

Convexity

For $\forall \phi_1^1(\delta), \phi_1^2(\delta) \in \Phi_{S1}$ and $\forall \lambda \in (0, 1)$, define the new function $\hat{\phi}(\delta) = \lambda * \phi_1^1(\delta) + (1 - \lambda) * \phi_1^2(\delta)$, it is trival that $\hat{\phi}(\delta) \in C^1[0, 1]$, $0 \leq \hat{\phi}(\delta) \leq \lambda + 1 - \lambda = 1$ and $\hat{\phi}'(\delta) = \lambda * \phi_1^{1\prime}(\delta) + (1 - \lambda) * \phi_1^{2\prime}(\delta) > 0$. So $\hat{\phi}(\delta) \in \Phi_{S1}$ for $\forall \lambda \in (0, 1)$.

Boundedness

By definition of normed linear space Φ_{S1}, it is trival that Φ_{S1} is bounded.

Equicontinuity

We already proved the boundedness of $\frac{d\overline{\triangle V}(\delta)}{d\delta}$ and thus the boundedness of $\overline{\lambda_1^*}'(\delta) = \frac{d\overline{\lambda_1^*}(\delta)}{d\delta} = \frac{-1}{2c_1} \frac{d\overline{\triangle V}(\delta)}{d\delta} \int_0^{1-\delta} \frac{\overline{\lambda_1^*}(\delta')}{\Lambda_1} \phi_1(\delta') d\delta'$. Next we need to prove the boundedness of $\frac{d\phi_1(\delta)}{d\delta}$.

$$
\begin{aligned}
&\frac{d\phi_1(\delta)}{d\delta} \\
&= \frac{d}{d\delta} \left[\frac{\alpha \int_0^1 \lambda_1^*(\delta')\phi_1(\delta')d\delta' + 2\lambda_1^*(1-\delta) \int_0^{\delta} \lambda_1^*(\delta')\phi_1(\delta')d\delta'}{2\alpha \int_0^1 \lambda_1^*(\delta')\phi_1(\delta')d\delta' + 2\lambda_1^*(1-\delta) \int_0^{\delta} \lambda_1^*(\delta')\phi_1(\delta')d\delta' + 2\lambda_1^*(\delta) \int_0^{1-\delta} \lambda_1^*(\delta')\phi_1(\delta')d\delta'} \right] \\
&= \frac{\left[\frac{\lambda_1^*(1-\delta)\lambda_1^*(\delta)\phi_1(\delta)}{\Lambda_1} - \lambda_1^{*\prime}(1-\delta)a(\delta) \right] \left[\frac{\alpha}{2} + 2\lambda_1^*(\delta)b(\delta) \right]}{\left(\alpha + 2\lambda_1^*(1-\delta)a(\delta) + 2\lambda_1^*(\delta)b(\delta)\right)^2} \\
&- \frac{\left[\lambda_1^{*\prime}(\delta)b(\delta) - \frac{\lambda_1^*(1-\delta)\lambda_1^*(\delta)\phi_1(1-\delta)}{\Lambda_1} \right] \left[\frac{\alpha}{2} + 2\lambda_1^*(1-\delta)a(\delta) \right]}{\left(\alpha + 2\lambda_1^*(1-\delta)a(\delta) + 2\lambda_1^*(\delta)b(\delta)\right)^2}
\end{aligned}
$$

We already proved the boundedness of $\lambda_1^*(\delta)$, $\lambda_1^{*\prime}(\delta)$, $a(\delta)$, $b(\delta)$, and $\exists \hat{\epsilon} > 0$ s.t. $\hat{\epsilon} \leq \Lambda_1 \leq 2\lambda_1^*(0)$, then if we plug in the given fixed points $\overline{\lambda_1^*}(\delta)$ and $\overline{\triangle V}(\delta)$ into the above equation, we will obtain the boundedness of $\frac{d\phi_1(\delta)}{d\delta}$, denote $\underset{\delta \in [0,1]}{max} |\frac{d\phi_1(\delta)}{d\delta}| = B_{d\phi_1}$.

Then for $\forall \phi_1(\delta) \in \Phi_{S1}$ and $\forall \delta \in [0,1]$: given $\forall \epsilon > 0$, we can always choose small enough $\hat{\triangle} = \frac{\epsilon}{2B_{d\phi_1}} > 0$, such that,

$$|\phi_1(\delta + \hat{\triangle}) - \phi_1(\delta)| = \left| \frac{d\phi_1(\delta)}{d\delta} \hat{\triangle} + o(|\hat{\triangle}|) \right| \leq 2 * \hat{\triangle} * \left| \frac{d\phi_1(\delta)}{d\delta} \right| \leq 2 * \frac{\epsilon}{2B_{d\phi_1}} * B_{d\phi_1} = \epsilon. \quad (76)$$

Since $\hat{\triangle}$ does not relate to specific value of δ, then any sequence of functions in normed linear space Φ_{S1} is uniform equicontinuous on $[0,1]$. Based on **boundedness** and **equicontinuity** above, and refer to Arzelä-Ascoli theorem, we prove the continuous transformation T_3, under given fixed points $\overline{\lambda_1^*}(\delta)$ and $\overline{\triangle V}(\delta)$, maps Φ_{S1} to Φ_{S1}, where the normed linear space Φ_{S1} is nonempty, convex and compact. By Schauder's fixed point theorem, given fixed points $\overline{\lambda_1^*}(\delta)$ and $\overline{\triangle V}(\delta)$, there exists fixed point $\overline{\phi_1}(\delta)$ of $\phi_1(\delta)$.

By (1)-(3) above, we prove that there exists fixed points $\overline{\triangle V}(\delta)$, $\overline{\lambda_1^*}(\delta)$, $\overline{\phi_1}(\delta)$ for the transformation $T : \triangle V_S \times \Lambda_{S1} \times \Phi_{S1} \longrightarrow \triangle V_S \times \Lambda_{S1} \times \Phi_{S1}$ defined above, given any parameters $r > 0$, $\alpha > 0$ and $c_1 > 0$.

D Proposition 3

We firstly consider symmetric and convex $f_\delta(\delta)$:

$$f'_\delta(\delta) \begin{cases} < 0 & \forall \delta \in [0, \frac{1}{2}); \\ = 0 & \delta = \frac{1}{2}; \\ > 0 & \forall \delta \in (\frac{1}{2}, 1]. \end{cases}$$

and

$$f_\delta(\delta) = f_\delta(1-\delta), \quad \forall \delta \in [0,1]$$

$$f_\delta(\delta) = \phi_1(\delta) + \phi_0(\delta), \quad \forall \delta \in [0,1]$$

By equilibrium condition (13) with $\hat{f}_\delta(\delta) = f_\delta(\delta)$,
we obtain (using the notations $A(\delta), B(\delta), a(\delta), b(\delta)$ in Appendix B):

$$\begin{aligned}
&\frac{d\dot{\phi}_1(\delta)}{d\delta} \\
&= 0 \\
&= -\alpha\phi_1'(\delta) + \frac{\alpha}{2} f_\delta'(\delta) - 2\lambda_1^{*'}(\delta)\phi_1(\delta)b(\delta) - 2\lambda_1^*(\delta)\phi_1'(\delta)b(\delta) + 2\frac{\lambda_1^*(\delta)\phi_1(\delta)\lambda_0^*(\delta)\phi_0(\delta)}{\Lambda_0} \\
&+ 2\lambda_0^{*'}(\delta)\phi_0(\delta)a(\delta) + 2\lambda_0^*(\delta)(f_\delta'(\delta) - \phi_1'(\delta))a(\delta) + 2\frac{\lambda_1^*(\delta)\phi_1(\delta)\lambda_0^*(\delta)\phi_0(\delta)}{\Lambda_1}, \\
&\quad \forall\delta \in [0,1] \qquad (77)
\end{aligned}$$

since the sum of all the terms not including $f_\delta'(\delta)$ or $\phi_1'(\delta)$ is positive, then

$$-\left(\alpha + 2\lambda_1^*(\delta)b(\delta) + 2\lambda_0^*(\delta)a(\delta)\right)\phi_1'(\delta) + \left(\frac{\alpha}{2} + 2\lambda_0^*(\delta)a(\delta)\right) f_\delta'(\delta) < 0, \quad \forall\delta \in [0,1] \qquad (78)$$

By the definition and sign of $f_\delta'(\delta)$, we obtain

$$\phi_0'(\delta) < 0 \qquad \forall\delta \in [0, \frac{1}{2}) \qquad (79)$$

$$\phi_1'(\delta) > 0 \qquad \forall\delta \in (\frac{1}{2}, 1] \qquad (80)$$

$$\left(\frac{\alpha}{2} + 2\lambda_0^*(\delta)a(\delta)\right)\phi_0'(\delta) < \left(\frac{\alpha}{2} + 2\lambda_1^*(\delta)b(\delta)\right)\phi_1'(\delta) \qquad \forall\delta \in [0,1] \qquad (81)$$

and since $f_\delta'(\frac{1}{2}) = 0$

$$\phi_1'(\frac{1}{2}) = -\phi_0'(\frac{1}{2}) > 0 \qquad (82)$$

Suppose $\exists\delta_1^* \in [0, \frac{1}{2})$, s.t. $\phi_1'(\delta_1^*) < 0$ and $\nexists\delta^* \in [0,1]$ s.t.

$$\left(\frac{\alpha}{2} + 2\lambda_0^*(\delta^*)a(\delta^*)\right) f_\delta'(\delta^*) + \frac{1}{c_1}\frac{d\triangle V(\delta^*)}{d\delta}\left(a(\delta^*)^2\phi_0(\delta^*) + b(\delta^*)^2\phi_1(\delta^*)\right) \qquad (83)$$

$$+2\lambda_1^*(\delta^*)\lambda_0^*(\delta^*)\phi_1(\delta^*)\phi_0(\delta^*)\left(\frac{1}{\Lambda_0} + \frac{1}{\Lambda_1}\right) = 0$$

Suppose all equilibrium components are smooth, by Mean Value Theorem, $\exists \delta_2^* \in (\delta_1^*, \frac{1}{2})$ s.t. $\phi_1'(\delta_2^*) = 0$. By (77), we obtain

$$\left(\frac{\alpha}{2} + 2\lambda_0^*(\delta_2^*)a(\delta_2^*)\right) f_\delta'(\delta_2^*) + \frac{1}{c_1}\frac{d\triangle V(\delta_2^*)}{d\delta}\left(a(\delta_2^*)^2\phi_0(\delta_2^*) + b(\delta_2^*)^2\phi_1(\delta_2^*)\right) \tag{84}$$

$$+2\lambda_1^*(\delta_2^*)\lambda_0^*(\delta_2^*)\phi_1(\delta_2^*)\phi_0(\delta_2^*)\left(\frac{1}{\Lambda_0} + \frac{1}{\Lambda_1}\right) = 0$$

which contradicts with condition (83). Then we conclude $\nexists \delta_1^* \in [0, \frac{1}{2})$, s.t. $\phi_1'(\delta_1^*) < 0$. So if condition (83) is satisfied,

$$\phi_1'(\delta) > 0 \quad \forall \delta \in [0, 1] \tag{85}$$

Similar idea works for the sign of $\phi_0'(\delta)$ on $\delta \in (\frac{1}{2}, 1]$. Then we conclude as long as condition (83) applies,

$$\phi_0'(\delta) < 0 < \phi_1'(\delta) \quad \forall \delta \in [0, 1] \tag{86}$$

And the same conclusion applies when $f_\delta(\delta)$ is symmetric but concave.

E Proposition 4

We use the Implicit Function Theorem to show the effects of α and c_1 on all the competitive equilibrium components $\triangle V(\delta)$, $\lambda_1^*(\delta)$ and $\phi_1(\delta)$ on $\delta \in [0, 1]$. Since we only focus on symmetric equilibrium defined in Definition 3.2, we have the other two components as $\lambda_0^*(\delta) = \lambda_1^*(1-\delta)$ and $\phi_0(\delta) = \phi_1(1-\delta)$ for $\forall \delta \in [0, 1]$.

We write the three competitive equilibrium conditions collectively as follows:

$$H(\triangle V(\delta), \lambda_1^*(\delta), \phi_1(\delta); \alpha, c_1) = \begin{bmatrix} H_1(\triangle V(\delta), \lambda_1^*(\delta), \phi_1(\delta); \alpha, c_1) \\ H_2(\triangle V(\delta), \lambda_1^*(\delta), \phi_1(\delta); \alpha, c_1) \\ H_3(\triangle V(\delta), \lambda_1^*(\delta), \phi_1(\delta); \alpha, c_1) \end{bmatrix} \equiv 0_{3\times 1} \tag{87}$$

where

$$
\begin{aligned}
&H_1(\triangle V(\delta), \lambda_1^*(\delta), \phi_1(\delta); \alpha, c_1)\\
&= 2\alpha\phi_1(\delta)\int_0^1 \lambda_1^*(\delta)\phi_1(\delta)d\delta + 2\phi_1(\delta)\lambda_1^*(\delta)\int_0^{1-\delta}\lambda_1^*(\delta')\phi_1(\delta')d\delta'\\
&+ 2\phi_1(\delta)\lambda_1^*(1-\delta)\int_0^{\delta}\lambda_1^*(\delta')\phi_1(\delta')d\delta' - \alpha\int_0^1\lambda_1^*(\delta)\phi_1(\delta)d\delta - 2\lambda_1^*(1-\delta)\int_0^{\delta}\lambda_1^*(\delta')\phi_1(\delta')d\delta'\\
&\equiv 0 \qquad (88)
\end{aligned}
$$

$$
\begin{aligned}
&H_2(\triangle V(\delta), \lambda_1^*(\delta), \phi_1(\delta); \alpha, c_1)\\
&= (\alpha + r)\triangle V(\delta) - \delta - c_1\lambda_1^{*2}(\delta) + c_1\lambda_1^{*2}(1-\delta) - \alpha E[\triangle V(\delta)]\\
&= (\alpha + r)\triangle V(\delta) - \delta - c_1\lambda_1^{*2}(\delta) + c_1\lambda_1^{*2}(1-\delta) - \frac{\alpha}{2r}\\
&\equiv 0 \qquad (89)
\end{aligned}
$$

$$
\begin{aligned}
&H_3(\triangle V(\delta), \lambda_1^*(\delta), \phi_1(\delta); \alpha, c_1)\\
&= 2c_1\lambda_1^*(\delta) - (\triangle V(0) + \triangle V(1) - \triangle V(\delta))\int_0^{1-\delta}\frac{\lambda_1^*(\delta')\phi_1(\delta')}{\Lambda_1}d\delta'\\
&\quad + \int_0^{1-\delta}\frac{\lambda_1^*(\delta')\phi_1(\delta')\triangle V(\delta')}{\Lambda_1}d\delta'\\
&= 2c_1\lambda_1^*(\delta) - (\triangle V(0) + \triangle V(1) - \triangle V(\delta))\int_0^{1-\delta}F(\delta')d\delta' + \int_0^{1-\delta}F(\delta')\triangle V(\delta')d\delta'\\
&\equiv 0 \qquad (90)
\end{aligned}
$$

In the last but one equality, we use the notation $F(\delta) = \frac{\lambda_1^*(\delta)\phi_1(\delta)}{\Lambda_1}$ for simplicity.

By Implicit Function Theorem, we have the following general relation:
For any $i = 1, 2, 3$, any $\delta \in [0, 1]$ and any incrementals [19] $h_{\triangle V}(\delta), h_{\lambda_1^*}(\delta), h_{\phi_1}(\delta)$,

$$
\frac{\partial H_i}{\partial \triangle V(\delta)}h_{\triangle V}(\delta) + \frac{\partial H_i}{\partial \lambda_1^*(\delta)}h_{\lambda_1^*}(\delta) + \frac{\partial H_i}{\partial \phi_1(\delta)}h_{\phi_1}(\delta) + \frac{\partial H_i}{\partial c_1}\triangle c_1 \equiv 0 \qquad (91)
$$

[19] We will define the incrementals more formally in Section G.

$$\frac{\partial H_i}{\partial \triangle V(\delta)} h_{\triangle V}(\delta) + \frac{\partial H_i}{\partial \lambda_1^*(\delta)} h_{\lambda_1^*}(\delta) + \frac{\partial H_i}{\partial \phi_1(\delta)} h_{\phi_1}(\delta) + \frac{\partial H_i}{\partial \alpha} \triangle\alpha \equiv 0 \tag{92}$$

Specifically for $i = 1$, we have:

$$\frac{\partial H_1}{\partial \triangle V(\delta)} h_{\triangle V}(\delta) \equiv 0 \tag{93}$$

$$\begin{aligned}
&\frac{\partial H_1}{\partial \phi_1(\delta)} h_{\phi_1}(\delta) + \frac{\partial H_1}{\partial \lambda_1^*(\delta)} h_{\lambda_1^*}(\delta) \qquad (94)\\
&= \lim_{m\to 0} \Bigg\{ \frac{H_1(\lambda_1^*(\delta), \phi_1(\delta) + m h_{\phi_1}(\delta)) - H_1(\lambda_1^*(\delta), \phi_1(\delta))}{m} \\
&+ \frac{H_1(\lambda_1^*(\delta) + m h_{\lambda_1^*}(\delta), \phi_1(\delta)) - H_1(\lambda_1^*(\delta), \phi_1(\delta))}{m} \Bigg\} \\
&= 2\alpha\phi_1(\delta) \int_0^1 \lambda_1^*(\delta) h_{\phi_1}(\delta) d\delta + 2\alpha h_{\phi_1}(\delta) \int_0^1 \lambda_1^*(\delta)\phi_1(\delta) d\delta + 2\phi_1(\delta)\lambda_1^*(\delta) \int_0^{1-\delta} \lambda_1^*(\delta') h_{\phi_1}(\delta') d\delta' \\
&+ 2h_{\phi_1}(\delta)\lambda_1^*(\delta) \int_0^{1-\delta} \lambda_1^*(\delta')\phi_1(\delta') d\delta' + 2\phi_1(\delta)\lambda_1^*(1-\delta) \int_0^{\delta} \lambda_1^*(\delta') h_{\phi_1}(\delta') d\delta' \\
&+ 2h_{\phi_1}(\delta)\lambda_1^*(1-\delta) \int_0^{\delta} \lambda_1^*(\delta')\phi_1(\delta') d\delta' - \alpha \int_0^1 \lambda_1^*(\delta) h_{\phi_1}(\delta) d\delta - 2\lambda_1^*(1-\delta) \int_0^{\delta} \lambda_1^*(\delta') h_{\phi_1}(\delta') d\delta' \\
&+ 2\alpha\phi_1(\delta) \int_0^1 h_{\lambda_1^*}(\delta)\phi_1(\delta) d\delta + 2\phi_1(\delta) h_{\lambda_1^*}(\delta) \int_0^{1-\delta} \lambda_1^*(\delta')\phi_1(\delta') d\delta' \\
&+ 2\phi_1(\delta)\lambda_1^*(\delta) \int_0^{1-\delta} h_{\lambda_1^*}(\delta')\phi_1(\delta') d\delta' + 2\phi_1(\delta)\lambda_1^*(1-\delta) \int_0^{\delta} h_{\lambda_1^*}(\delta')\phi_1(\delta') d\delta' \\
&+ 2\phi_1(\delta) h_{\lambda_1^*}(1-\delta) \int_0^{\delta} \lambda_1^*(\delta')\phi_1(\delta') d\delta' - \alpha \int_0^1 h_{\lambda_1^*}(\delta)\phi_1(\delta) d\delta \\
&- 2\lambda_1^*(1-\delta) \int_0^{\delta} h_{\lambda_1^*}(\delta')\phi_1(\delta') d\delta' - 2h_{\lambda_1^*}(1-\delta) \int_0^{\delta} \lambda_1^*(\delta')\phi_1(\delta') d\delta' \\
&\equiv 0 \quad \text{on } \delta \in [0,1]
\end{aligned}$$

$$\frac{\partial H_1}{\partial c_1}(\delta)\triangle c_1 \equiv 0 \quad \text{and} \quad \frac{\partial H_1}{\partial \alpha}(\delta)\triangle\alpha = \left(2\phi_1(\delta) \int_0^1 \lambda_1^*(\delta)\phi_1(\delta) d\delta - \int_0^1 \lambda_1^*(\delta)\phi_1(\delta) d\delta\right) \triangle\alpha \tag{95}$$

Since by (95),

$$\frac{\partial H_1}{\partial \alpha}(\delta)\triangle\alpha + \frac{\partial H_1}{\partial \alpha}(1-\delta)\triangle\alpha = 0 \tag{96}$$

and by (91)(92),

$$
\begin{aligned}
&\frac{\partial H}{\partial \phi_1(\delta)} h_{\phi_1}(\delta) + \frac{\partial H}{\partial \lambda_1^*(\delta)} h_{\lambda_1^*}(\delta) + \frac{\partial H}{\partial \phi_1(\delta)} h_{\phi_1}(1-\delta) + \frac{\partial H}{\partial \lambda_1^*(\delta)} h_{\lambda_1^*}(1-\delta) \\
&= 2(h_{\phi_1}(\delta) + h_{\phi_1}(1-\delta)) \left(\alpha \int_0^1 \lambda_1^*(\delta')\phi_1(\delta')d\delta' + \lambda_1^*(\delta) \int_0^{1-\delta} \lambda_1^*(\delta')\phi_1(\delta')d\delta' \right. \\
&\left. + \lambda_1^*(1-\delta) \int_0^{\delta} \lambda_1^*(\delta')\phi_1(\delta')d\delta' \right) \\
&\equiv 0 \qquad (97)
\end{aligned}
$$

we obtain that for either changing α or changing c_1:

$$
h_{\phi_1}(\delta) + h_{\phi_1}(1-\delta) \equiv 0, \quad \forall \delta \in [0,1] \qquad (98)
$$

Specifically for $i = 2$:

$$
\begin{aligned}
&(\alpha + r) h_{\triangle V}(\delta) + 2c_1 \left(\lambda_1^*(1-\delta) h_{\lambda_1^*}(1-\delta) - \lambda_1^*(\delta) h_{\lambda_1^*}(\delta) \right) + \triangle c_1 \left(\lambda_1^{*2}(1-\delta) - \lambda_1^{*2}(\delta) \right) \\
&\equiv 0, \quad \forall \delta \in [0,1] \qquad (99)
\end{aligned}
$$

and

$$
\begin{aligned}
&(\alpha + r) h_{\triangle V}(\delta) + 2c_1 \left(\lambda_1^*(1-\delta) h_{\lambda_1^*}(1-\delta) - \lambda_1^*(\delta) h_{\lambda_1^*}(\delta) \right) + \triangle \alpha \left(\triangle V(\delta) - \frac{1}{2r} \right) \\
&\equiv 0, \quad \forall \delta \in [0,1] \qquad (100)
\end{aligned}
$$

By (99)(100), we can also plug in $1-\delta$ without changing the equalities. Then we obtain that for either changing α or changing c_1:

$$
h_{\triangle V}(\delta) + h_{\triangle V}(1-\delta) \equiv 0, \quad \forall \delta \in [0,1] \qquad (101)
$$

(98)(101) further give us:

$$
h'_{\phi_1}(\delta) = h'_{\phi_1}(1-\delta), \quad \forall \delta \in [0,1] \qquad (102)
$$

$$
h'_{\triangle V}(\delta) = h'_{\triangle V}(1-\delta), \quad \forall \delta \in [0,1] \qquad (103)
$$

$$h_{\triangle V}(\frac{1}{2}) = h_{\phi_1}(\frac{1}{2}) = 0 \tag{104}$$

Then as long as we can identify the sign of $h'_{\phi_1}(\delta)$ (or $h'_{\triangle V}(\delta)$) for any $\delta \in [0,1]$, then it will be sufficient to characterize the change in the shape of asset-owner density $h_{\phi_1}(\delta)$ on the whole interval. Here we specifically focus on the utility type $\delta = 0$, by condition H_1:

$$\phi_1(0) = \frac{\frac{\alpha}{2}}{\alpha + \lambda_1^*(0)} \tag{105}$$

By condition that if c_1 increases, $h_{\lambda_1^*}(0) < 0$, then it is trival by (105) that $h_{\phi_1}(0) > 0$;
By condition that if α increases, $h_{\lambda_1^*}(0) < 0$, then by (105):

$$(\phi_1(0) - \frac{1}{2})\triangle\alpha + (\alpha + \lambda_1^*(0))h_{\phi_1}(0) + \phi_1(0)h_{\lambda_1^*}(0) = 0 \tag{106}$$

since the first and third terms in (106) are both negative, then

$$h_{\phi_1}(0) > 0 \tag{107}$$

If (107) applies when c_1 and/or α increases, then by (102)(104), it is trival to prove by contradiction that $h'_{\phi_1}(\delta) < 0, \quad \forall\delta \in [0,1]$. □

F Proposition 5

F.1

For symmetric equilibrium with $f_\delta(\delta) \equiv 1$, $\forall\delta \in [0,1]$,

$$\bar{\lambda}(\delta) = \phi_1(\delta)\lambda_1^*(\delta) + \phi_0(\delta)\lambda_0^*(\delta) \tag{108}$$

Using the notations $A(\delta), B(\delta), a(\delta), b(\delta)$ in Appendix B,

$$\lambda_1^{*\prime}(\delta) = -\frac{1}{2c_1}\frac{d\triangle V(\delta)}{d\delta}b(\delta) \tag{109}$$

$$\lambda_0^{*\prime}(\delta) = \frac{1}{2c_1}\frac{d\triangle V(\delta)}{d\delta}a(\delta) \tag{110}$$

$$\phi_0'(\delta) = \frac{2\left[(\lambda_1^*(\delta)b(\delta))'(\frac{\alpha}{2} + 2\lambda_0^*(\delta)a(\delta)) - (\lambda_0^*(\delta)a(\delta))'(\frac{\alpha}{2} + 2\lambda_1^*(\delta)b(\delta))\right]}{(\alpha + 2\lambda_0^*(\delta)a(\delta) + 2\lambda_1^*(\delta)b(\delta))^2} \tag{111}$$

$$\phi_1'(\delta) = -\phi_0'(\delta) = \frac{2\left[(\lambda_0^*(\delta)a(\delta))'(\frac{\alpha}{2} + 2\lambda_1^*(\delta)b(\delta)) - (\lambda_1^*(\delta)b(\delta))'(\frac{\alpha}{2} + 2\lambda_0^*(\delta)a(\delta)))\right]}{(\alpha + 2\lambda_0^*(\delta)a(\delta) + 2\lambda_1^*(\delta)b(\delta))^2} \tag{112}$$

$\Longrightarrow$

$$\lambda_1^{*\prime}(0) = -\frac{1}{4c_1}\frac{1}{r + \alpha + \frac{\lambda_1^*(0)}{2}} \tag{113}$$

$$\lambda_0^{*\prime}(0) = 0 \tag{114}$$

$$\begin{aligned}\phi_0'(0) &= \frac{\alpha(\lambda_1^*(\delta)b(\delta))'|_{\delta=0} - (\lambda_0^*(\delta)a(\delta))'|_{\delta=0}(\alpha + 2\lambda_1^*(0))}{(\alpha + \lambda_1^*(0))^2} \\ &= \frac{-\frac{\alpha}{8c_1}\frac{1}{r+\alpha+\frac{\lambda_1^*(0)}{2}}}{(\alpha + \lambda_1^*(0))^2}\end{aligned} \tag{115}$$

$\Longrightarrow$

$$\begin{aligned}\bar{\lambda}'(0) &= \phi_1'(0)\lambda_1^*(0) + \phi_1(0)\lambda_1^{*\prime}(0) \\ &= \frac{-\frac{\alpha}{8c_1}\frac{\lambda_1^*(0)}{r+\alpha+\frac{\lambda_1^*(0)}{2}}}{(\alpha + \lambda_1^*(0))^2} - \frac{1}{4c_1}\frac{\phi_1(0)}{r + \alpha + \frac{\lambda_1^*(0)}{2}} \\ &< 0\end{aligned} \tag{116}$$

By symmetry,

$$\begin{aligned}&\bar{\lambda}'(1-\delta) \\ &= \phi_1'(1-\delta)\lambda_1^*(1-\delta) + \phi_1(1-\delta)\lambda_1^{*\prime}(1-\delta) + \phi_0'(1-\delta)\lambda_0^*(1-\delta) + \phi_0(1-\delta)\lambda_0^{*\prime}(1-\delta) \\ &= -\phi_0'(\delta)\lambda_0^*(\delta) - \phi_0(\delta)\lambda_0^{*\prime}(\delta) - \phi_1'(\delta)\lambda_1^*(\delta) - \phi_1(\delta)\lambda_1^{*\prime}(\delta) \\ &= -\bar{\lambda}'(\delta)\end{aligned} \tag{117}$$

then

$$\bar{\lambda}'(1) = -\bar{\lambda}'(0) > 0 \tag{118}$$

and

$$\bar{\lambda}'(\frac{1}{2}) = -\bar{\lambda}'(1-\frac{1}{2}) = -\bar{\lambda}'(\frac{1}{2}) \tag{119}$$

$\Longrightarrow$

$$\bar{\lambda}'(\frac{1}{2}) = 0 \tag{120}$$

F.2

Lemma 1: *(1) As $c_1 \to +\infty$: $\lambda_1^*(\delta) \to 0$ and $\lambda_0^*(\delta) \to 0$ for $\forall \delta \in (0,1)$; (2) As $c_1 \to 0$: given $\forall \hat{\delta} \in (0,1)$ and $\forall M > 0$, $\lambda_1^*(\hat{\delta}) > M$ and $\lambda_0^*(\hat{\delta}) > M$.*

Proof:

By boundedness of $\triangle V(\delta)$ in (72), we have for $\forall \delta \in (0,1)$

$$0 < \int_{\delta}^{1} \frac{\lambda_0^*(\delta')}{\Lambda_0}(\triangle V(\delta')-\triangle V(\delta))\phi_0(\delta')d\delta' < (\triangle V(1)-\triangle V(0)) \int_{\delta}^{1} \frac{\lambda_0^*(\delta')}{\Lambda_0}\phi_0(\delta')d\delta' < \frac{1}{2(\alpha+r)} \tag{121}$$

$$0 < \int_{0}^{\delta} \frac{\lambda_1^*(\delta')}{\Lambda_1}(\triangle V(\delta)-\triangle V(\delta'))\phi_1(\delta')d\delta' < (\triangle V(1)-\triangle V(0)) \int_{0}^{\delta} \frac{\lambda_1^*(\delta')}{\Lambda_1}\phi_1(\delta')d\delta' < \frac{1}{2(\alpha+r)} \tag{122}$$

then it is trival that for any fixed α, as $c_1 \to +\infty$,

$$\lambda_1^*(\delta) = \frac{\int_{\delta}^{1} \frac{\lambda_0^*(\delta')}{\Lambda_0}(\triangle V(\delta') - \triangle V(\delta))\phi_0(\delta')d\delta'}{2c_1} \to 0 \tag{123}$$

$$\lambda_0^*(\delta) = \frac{\int_{0}^{\delta} \frac{\lambda_1^*(\delta')}{\Lambda_1}(\triangle V(\delta) - \triangle V(\delta'))\phi_1(\delta')d\delta'}{2c_1} \to 0 \tag{124}$$

By symmetry,

$$\lambda_1^*(\delta) = \frac{\int_{0}^{1-\delta} \frac{\lambda_1^*(\delta')}{\Lambda_1}(\triangle V(1-\delta') - \triangle V(\delta))\phi_1(\delta')d\delta'}{2c_1} \to 0 \tag{125}$$

and also by Section C, $\lambda_1^*(\delta) \in \Lambda_{S1}$, $\phi_1(\delta) \in \Phi_{S1}$ and $\triangle V(\delta) \in \triangle V_S$ where Λ_{S1}, Φ_{S1} and $\triangle V_S$ are compact sets. Then for each fixed $\hat{\delta} \in (0,1)$, by Extreme Value Theorem,

$\exists(\lambda_{1S}^*(\delta), \phi_{1S}(\delta), \triangle V_S(\delta))$ s.t.

$$
\begin{aligned}
&(\lambda_{1S}^*(\delta), \phi_{1S}(\delta), \triangle V_S(\delta)) \\
&= \underset{(\lambda_1^*(\delta),\phi_1(\delta),\triangle V(\delta))\in\Lambda_{S1}\times\Phi_{S1}\times\triangle V_S}{argmax} \left\{\int_0^{1-\delta} \frac{\lambda_1^*(\delta')}{\Lambda_1}(\triangle V(\delta) - \triangle V(1-\delta'))\phi_1(\delta')d\delta'\right\}
\end{aligned} \tag{126}
$$

and

$$
\begin{aligned}
&M^* \\
&= \underset{(\lambda_1^*(\delta),\phi_1(\delta),\triangle V(\delta))\in\Lambda_{S1}\times\Phi_{S1}\times\triangle V_S}{max} \left\{\int_0^{1-\delta} \frac{\lambda_1^*(\delta')}{\Lambda_1}(\triangle V(\delta) - \triangle V(1-\delta'))\phi_1(\delta')d\delta'\right\}
\end{aligned} \tag{127}
$$

then for any other large constant $M > 0$, we can always find $c_1^*(\hat{\delta}) = -\frac{M^*}{2M}$ s.t.

$$
\lambda_1^*(\hat{\delta}) = -\frac{\int_0^{1-\hat{\delta}} \frac{\lambda_1^*(\delta')}{\Lambda_1}(\triangle V(\delta) - \triangle V(1-\delta'))\phi_1(\delta')d\delta'}{2c_1} > M \quad \forall c_1 < c_1^*(\hat{\delta}) \tag{128}
$$

i.e. for each fixed $\hat{\delta} \in (0,1)$,

$$
\lambda_1^*(\hat{\delta}) \to +\infty \quad \text{as } c_1 \to 0 \tag{129}
$$

□

$$
\begin{aligned}
\bar{\lambda}'(\delta) &= \phi_1'(\delta)\lambda_1^*(\delta) + \phi_1(\delta)\lambda_1^{*\prime}(\delta) + \phi_0'(\delta)\lambda_0^*(\delta) + \phi_0(\delta)\lambda_0^{*\prime}(\delta) &(130)\\
&= \phi_1'(\delta)(\lambda_1^*(\delta) - \lambda_0^*(\delta)) + \frac{1}{2c_1}\frac{d\triangle V(\delta)}{d\delta}(\phi_0(\delta)a(\delta) - \phi_1(\delta)b(\delta)) &(131)
\end{aligned}
$$

where

$$
\begin{aligned}
\phi_1'(\delta) =& \frac{(\alpha + 4\lambda_1^*(\delta)b(\delta))\left(\frac{a^2(\delta)}{2c_1}\frac{d\triangle V(\delta)}{d\delta} + \frac{\lambda_0^*(\delta)\lambda_1^*(\delta)\phi_1(\delta)}{\Lambda}\right)}{(\alpha + 2\lambda_0^*(\delta)a(\delta) + 2\lambda_1^*(\delta)b(\delta))^2} \\
&+ \frac{(\alpha + 4\lambda_0^*(\delta)a(\delta))\left(\frac{b^2(\delta)}{2c_1}\frac{d\triangle V(\delta)}{d\delta} + \frac{\lambda_0^*(\delta)\lambda_1^*(\delta)\phi_0(\delta)}{\Lambda}\right)}{(\alpha + 2\lambda_0^*(\delta)a(\delta) + 2\lambda_1^*(\delta)b(\delta))^2}
\end{aligned} \tag{132}
$$

$$
\Lambda = \Lambda_1 = \Lambda_0 \tag{133}
$$

(1) For each α, by Lemma 1,

$$\begin{aligned}
&\bar{\lambda}'(\delta) \\
&= \frac{1}{2c_1}\left\{2c_1\phi_1'(\delta)(\lambda_1^*(\delta)-\lambda_0^*(\delta)) + \frac{d\triangle V(\delta)}{d\delta}(\phi_0(\delta)a(\delta)-\phi_1(\delta)b(\delta))\right\}
\end{aligned} \tag{134}$$

where

$$\begin{aligned}
&\lim_{c_1\to+\infty} 2c_1\phi_1'(\delta) \\
&= \lim_{c_1\to+\infty}\left\{\frac{(\alpha+4\lambda_1^*(\delta)b(\delta))\left(a^2(\delta)\frac{d\triangle V(\delta)}{d\delta}+\frac{\lambda_0^*(\delta)2c_1\lambda_1^*(\delta)\phi_1(\delta)}{\Lambda}\right)}{(\alpha+2\lambda_0^*(\delta)a(\delta)+2\lambda_1^*(\delta)b(\delta))^2}\right. \\
&\left. + \frac{(\alpha+4\lambda_0^*(\delta)a(\delta))\left(b^2(\delta)\frac{d\triangle V(\delta)}{d\delta}+\frac{\lambda_0^*(\delta)2c_1\lambda_1^*(\delta)\phi_0(\delta)}{\Lambda}\right)}{(\alpha+2\lambda_0^*(\delta)a(\delta)+2\lambda_1^*(\delta)b(\delta))^2}\right\} \\
&= \frac{0}{\alpha^2} \\
&= 0 \quad \forall\delta\in(0,\frac{1}{2})
\end{aligned} \tag{135}$$

$$\lim_{c_1\to+\infty}\frac{\phi_0(\delta)}{\phi_1(\delta)} = \lim_{c_1\to+\infty}\frac{\frac{\alpha}{2}+2\lambda_1^*(\delta)b(\delta)}{\frac{\alpha}{2}+2\lambda_0^*(\delta)a(\delta)} = 1 < \frac{b(\delta)}{a(\delta)} \quad \forall\delta\in(0,\frac{1}{2}) \tag{136}$$

and notations $a(\delta)$ and $b(\delta)$ follow Section B.
Then (135)(136) and "$\bar{\lambda}'(0)<0$" $\Longrightarrow$

$$\lim_{c_1\to+\infty}\bar{\lambda}'(\delta) < 0 \quad \forall\delta\in[0,\frac{1}{2}) \tag{137}$$

We also assume that

$$\lambda_1^*(\delta_1;c_1) = \Omega\left(\lambda_1^*(\delta_2;c_1)\right)(c_1\to+\infty) \quad \forall\delta_1,\delta_2\in[0,\frac{1}{2}] \tag{138}$$

$$\int_0^{1-\delta}\lambda_1^*(\delta';c_1)(\triangle V(\delta)-\triangle V(1-\delta'))\phi_1(\delta')d\delta' = \Omega(\Lambda_1(c_1))(c_1\to+\infty) \quad \forall\delta\in(0,1) \tag{139}$$

which are the negation of $\lambda_1^*(\delta_1;c_1) = o\left(\lambda_1^*(\delta_2;c_1)\right)(c_1\to+\infty)$ and $c_1\lambda_1^*(\delta;c_1) = o(1)(c_1\to+\infty)$.

Then

$$\bar{\lambda}(0) = \frac{\alpha \lambda_1^*(0)}{\alpha + \lambda_1^*(0)} \tag{140}$$

$$\bar{\lambda}(\frac{1}{2}) = \lambda_1^*(\frac{1}{2}) \tag{141}$$

then by (138) and Lemma 1,

$$\lim_{c_1 \to +\infty} \frac{\bar{\lambda}(0)}{\bar{\lambda}(\frac{1}{2})} = \lim_{c_1 \to +\infty} \frac{\alpha \frac{\lambda_1^*(0)}{\lambda_1^*(\frac{1}{2})}}{\alpha + \lambda_1^*(0)} = \frac{\lambda_1^*(0)}{\lambda_1^*(\frac{1}{2})} > 1 \tag{142}$$

Then we calculate that,

$$\begin{aligned} \bar{\lambda}''(\frac{1}{2}) &= \frac{\frac{d\triangle V(\frac{1}{2})}{d\delta}}{2c_1^2 \Lambda_1(\alpha + 4\lambda_1^*(\frac{1}{2})b(\frac{1}{2}))} \\ &\times \underbrace{\left\{ -4\frac{d\triangle V(\frac{1}{2})}{d\delta} b(\frac{1}{2}) \left(\int_0^{\frac{1}{2}} \lambda_1^*(\delta')\phi_1(\delta')d\delta' \right) + \frac{c_1 \lambda_1^*(\frac{1}{2})\alpha}{2} - 2\lambda_1^*(\frac{1}{2})c_1\lambda_0^*(\frac{1}{2})b(\frac{1}{2}) \right\}}_{*} \end{aligned} \tag{143}$$

so the sign of $\bar{\lambda}''(\frac{1}{2})$ depends on the sign of the $*$ term in (143).
As by Lemma 1 and (139)

$$\begin{aligned} &\lim_{c_1 \to +\infty} \left\{ -4\frac{d\triangle V(\frac{1}{2})}{d\delta} b(\frac{1}{2}) \left(\int_0^{\frac{1}{2}} \lambda_1^*(\delta')\phi_1(\delta')d\delta' \right) + \frac{c_1 \lambda_1^*(\frac{1}{2})\alpha}{2} - 2\lambda_1^*(\frac{1}{2})c_1\lambda_0^*(\frac{1}{2})b(\frac{1}{2}) \right\} \\ &= \lim_{c_1 \to +\infty} \frac{c_1 \lambda_1^*(\frac{1}{2})\alpha}{2} > 0 \end{aligned} \tag{144}$$

then

$$\lim_{c_1 \to +\infty} \bar{\lambda}''(\frac{1}{2}) > 0 \tag{145}$$

Then by (137)(142)(145), we conclude that for each fixed α, $\exists c_1^1(\alpha), c_1^2(\alpha), c_1^3(\alpha)$ s.t.

$$\bar{\lambda}'(\delta) < 0 \quad \forall \delta \in [0, \frac{1}{2}) \quad \text{for } \forall c_1 > c_1^1(\alpha) \tag{146}$$

$$\bar{\lambda}(0) > \bar{\lambda}(\frac{1}{2}) \quad \text{for } \forall c_1 > c_1^2(\alpha) \tag{147}$$

$$\bar{\lambda}''(\frac{1}{2}) > 0 \quad \text{for } \forall c_1 > c_1^3(\alpha) \tag{148}$$

Then

$$c_1^*(\alpha) = max\{c_1^1(\alpha), c_1^2(\alpha), c_1^3(\alpha)\} \tag{149}$$

(2) For each c_1, since the following components are bounded:

$$0 < \frac{d\triangle V(\delta)}{d\delta} < \frac{1}{r+\alpha} \quad \forall \delta \in [0,1] \tag{150}$$

$$0 \leq \lambda_1^*(\delta) < \frac{1}{2c_1(\alpha+r)} \quad \forall \delta \in [0,1] \tag{151}$$

$$0 \leq a(\delta) \leq \frac{1}{2} \quad \forall \delta \in [0,1] \tag{152}$$

$$0 \leq b(\delta) \leq \frac{1}{2} \quad \forall \delta \in [0,1] \tag{153}$$

$\Longrightarrow$

$$\begin{aligned}
\lim_{\alpha\to+\infty} \bar{\lambda}'(\delta) &= \lim_{\alpha\to+\infty} \left\{ \phi_1'(\delta)(\lambda_1^*(\delta) - \lambda_0^*(\delta)) + \frac{1}{2c_1}\frac{d\triangle V(\delta)}{d\delta}(\phi_0(\delta)a(\delta) - \phi_1(\delta)b(\delta)) \right\} \qquad (154) \\
&= \lim_{\alpha\to+\infty} \frac{\frac{a^2(\delta)}{2c_1}\frac{d\triangle V(\delta)}{d\delta} + \frac{\lambda_0^*(\delta)\lambda_1^*(\delta)\phi_1(\delta)}{\Lambda} + \frac{b^2(\delta)}{2c_1}\frac{d\triangle V(\delta)}{d\delta} + \frac{\lambda_0^*(\delta)\lambda_1^*(\delta)\phi_0(\delta)}{\Lambda}}{\alpha} + \\
&\quad \underbrace{\lim_{\alpha\to+\infty} \frac{1}{2c_1}\frac{d\triangle V(\delta)}{d\delta}(\phi_0(\delta)a(\delta) - \phi_1(\delta)b(\delta))}_{**} \\
&= \underbrace{\lim_{\alpha\to+\infty} \frac{1}{2c_1}\frac{d\triangle V(\delta)}{d\delta}(\phi_0(\delta)a(\delta) - \phi_1(\delta)b(\delta))}_{**}
\end{aligned}$$

since

$$\lim_{\alpha\to+\infty} \frac{\phi_0(\delta)}{\phi_1(\delta)} = \lim_{\alpha\to+\infty} \frac{\frac{\alpha}{2} + 2\lambda_1^*(\delta)b(\delta)}{\frac{\alpha}{2} + 2\lambda_0^*(\delta)a(\delta)} = 1 < \frac{b(\delta)}{a(\delta)} \quad \forall \delta \in (0, \frac{1}{2}) \tag{155}$$

so

$$\lim_{\alpha\to+\infty} \bar{\lambda}'(\delta) = \underbrace{\lim_{\alpha\to+\infty} \frac{1}{2c_1}\frac{d\triangle V(\delta)}{d\delta}(\phi_0(\delta)a(\delta) - \phi_1(\delta)b(\delta))}_{**} < 0 \quad \forall \delta \in (0, \frac{1}{2}) \tag{156}$$

And it is trival that

$$\lim_{\alpha\to+\infty}\frac{\bar{\lambda}(0)}{\bar{\lambda}(\frac{1}{2})}=\lim_{\alpha\to+\infty}\frac{\alpha\frac{\lambda_1^*(0)}{\lambda_1^*(\frac{1}{2})}}{\alpha+\lambda_1^*(0)}=\frac{\lambda_1^*(0)}{\lambda_1^*(\frac{1}{2})}>1 \tag{157}$$

$$\lim_{\alpha\to+\infty}\bar{\lambda}''(\frac{1}{2})=\lim_{\alpha\to+\infty}\frac{\lambda_1^*(\frac{1}{2})\frac{d\triangle V(\frac{1}{2})}{d\delta}}{4c_1\Lambda_1}>0 \tag{158}$$

Then by (156)(157)(158), we conclude that for each fixed c_1, $\exists\alpha^1(c_1),\alpha^2(c_1),\alpha^3(c_1)$ s.t.

$$\bar{\lambda}'(\delta)<0 \quad \forall\delta\in[0,\frac{1}{2}) \quad \text{for } \forall\alpha>\alpha^1(c_1) \tag{159}$$

$$\bar{\lambda}(0)>\bar{\lambda}(\frac{1}{2}) \quad \text{for } \forall\alpha>\alpha^2(c_1) \tag{160}$$

$$\bar{\lambda}''(\frac{1}{2})>0 \quad \text{for } \forall\alpha>\alpha^3(c_1) \tag{161}$$

Then

$$\alpha^*(c_1)=max\{\alpha^1(c_1),\alpha^2(c_1),\alpha^3(c_1)\} \tag{162}$$

F.3

$$\bar{\lambda}'(\delta)=\underbrace{\phi_1'(\delta)(\lambda_1^*(\delta)-\lambda_0^*(\delta))}_{3*}+\underbrace{\frac{1}{2c_1}\frac{d\triangle V(\delta)}{d\delta}}_{4*}(\phi_0(\delta)a(\delta)-\phi_1(\delta)b(\delta)) \tag{163}$$

The terms 3∗ and 4∗ are always positive, so we only focus on the sign of $\phi_0(\delta)a(\delta)-\phi_1(\delta)b(\delta)$.

(1) For each α, by Lemma 1, $\exists\hat{\delta}\in(0,\frac{1}{2})$ s.t.

$$\lim_{c_1\to0}\frac{\phi_0(\delta)}{\phi_1(\delta)}=\lim_{c_1\to0}\frac{2\lambda_1^*(\delta)b(\delta)}{2\lambda_0^*(\delta)a(\delta)}>\frac{b(\delta)}{a(\delta)} \quad \forall\hat{\delta}<\delta<\frac{1}{2} \tag{164}$$

where the inequality "$>$" is by $\lambda_1^*(\delta)>\lambda_0^*(\delta)$ for $\forall\delta\in(0,\frac{1}{2})$ and $0<a(\hat{\delta})<a(\delta)$.
Then $\exists\hat{\delta}\in(0,\frac{1}{2})$ s.t.

$$\lim_{c_1\to0}\bar{\lambda}'(\delta)>0 \quad \forall\hat{\delta}<\delta<\frac{1}{2} \tag{165}$$

And we also assume that

$$\lambda_1^*(\delta_1; c_1) = \Omega\left(\lambda_1^*(\delta_2; c_1)\right)(c_1 \to 0) \quad \forall \delta_1, \delta_2 \in [0, \frac{1}{2}] \tag{166}$$

which is the negation of $\lambda_1^*(\delta_1; c_1) = o\left(\lambda_1^*(\delta_2; c_1)\right)(c_1 \to 0)$.
Then

$$\underset{c_1 \to 0}{lim} \frac{\bar{\lambda}(0)}{\bar{\lambda}(\frac{1}{2})} = \underset{c_1 \to 0}{lim} \frac{\alpha \frac{\lambda_1^*(0)}{\lambda_1^*(\frac{1}{2})}}{\alpha + \lambda_1^*(0)} = 0 < 1 \tag{167}$$

Since

$$\begin{aligned} 0 &< c_1 \lambda_1^*(\frac{1}{2}) \\ &= \frac{1}{2} \int_{\frac{1}{2}}^{1} \frac{\lambda_0^*(\delta')}{\Lambda_0} (\triangle V(\delta') - \triangle V(\frac{1}{2})) \phi_0(\delta') d\delta' \\ &< (\triangle V(1) - \triangle V(0)) \frac{1}{2} \int_{\frac{1}{2}}^{1} \frac{\lambda_0^*(\delta')}{\Lambda_0} \phi_0(\delta') d\delta' \\ &< \frac{1}{4(\alpha + r)} \end{aligned} \tag{168}$$

and by Lemma 1

$$\underset{c_1 \to 0}{lim} \lambda_0^*(\frac{1}{2}) = +\infty \tag{169}$$

then the dominant term in term "*" of (143) is "$-2\lambda_1^*(\frac{1}{2}) c_1 \lambda_0^*(\frac{1}{2}) b(\frac{1}{2})$".
So we have

$$\underset{c_1 \to 0}{lim} \bar{\lambda}''(\frac{1}{2}) = \underset{c_1 \to 0}{lim} \frac{\frac{d \triangle V(\frac{1}{2})}{d\delta}}{2 c_1^2 \Lambda_1 (\alpha + 4\lambda_1^*(\frac{1}{2}) b(\frac{1}{2}))} \left\{ -2\lambda_1^*(\frac{1}{2}) c_1 \lambda_0^*(\frac{1}{2}) b(\frac{1}{2}) \right\} < 0 \tag{170}$$

Then by (165)(167)(170), we conclude that for each fixed α, $\exists c_1^4(\alpha), c_1^5(\alpha), c_1^6(\alpha)$ s.t. $\exists \hat{\delta} \in (0, \frac{1}{2})$ s.t.

$$\bar{\lambda}'(\delta) > 0 \quad \forall \hat{\delta} < \delta < \frac{1}{2} \quad \text{for } \forall c_1 < c_1^4(\alpha) \tag{171}$$

$$\bar{\lambda}(0) < \bar{\lambda}(\frac{1}{2}) \quad \text{for } \forall c_1 < c_1^5(\alpha) \tag{172}$$

$$\bar{\lambda}''(\frac{1}{2}) < 0 \quad \text{for } \forall c_1 < c_1^6(\alpha) \tag{173}$$

Then

$$c_1^{**}(\alpha) = max\{c_1^4(\alpha), c_1^5(\alpha), c_1^6(\alpha)\} \tag{174}$$

(2) For each c_1, similar to the case of "fixed α", to discuss the sign of $\bar{\lambda}'(\delta)$, we only focus on the sign of $\phi_0(\delta)a(\delta) - \phi_1(\delta)b(\delta)$.
$\exists\hat{\delta} \in (0, \frac{1}{2})$ s.t.

$$\lim_{\alpha\to 0}\frac{\phi_0(\delta)}{\phi_1(\delta)} = \lim_{\alpha\to 0}\frac{2\lambda_1^*(\delta)b(\delta)}{2\lambda_0^*(\delta)a(\delta)} > \frac{b(\delta)}{a(\delta)} \quad \forall\hat{\delta} < \delta < \frac{1}{2} \tag{175}$$

where the inequality ">" is by $\lambda_1^*(\delta) > \lambda_0^*(\delta)$ for $\forall\delta \in (0, \frac{1}{2})$ and $0 < a(\hat{\delta}) < a(\delta)$.
Then $\exists\hat{\delta} \in (0, \frac{1}{2})$ s.t.

$$\lim_{\alpha\to 0}\bar{\lambda}'(\delta) > 0 \quad \forall\hat{\delta} < \delta < \frac{1}{2} \tag{176}$$

To compare $\bar{\lambda}(0)$ and $\bar{\lambda}(\frac{1}{2})$, similarly

$$\lim_{\alpha\to 0}\frac{\bar{\lambda}(0)}{\bar{\lambda}(\frac{1}{2})} = \lim_{\alpha\to 0}\frac{\alpha\frac{\lambda_1^*(0)}{\lambda_1^*(\frac{1}{2})}}{\alpha + \lambda_1^*(0)} = 0 < 1 \tag{177}$$

Also, as $\alpha \to 0$, the term "$\frac{c_1\lambda_1^*(\frac{1}{2})\alpha}{2} \to 0$" in "$*$" term of (143), so it is trival that

$$\lim_{\alpha\to 0}\bar{\lambda}''(\frac{1}{2}) < 0 \tag{178}$$

Then by (176)(177)(178), we conclude that for each fixed c_1, $\exists\alpha^4(c_1), \alpha^5(c_1), \alpha^6(c_1)$ s.t.
$\exists\hat{\delta} \in (0, \frac{1}{2})$ s.t.

$$\bar{\lambda}'(\delta) > 0 \quad \forall\hat{\delta} < \delta < \frac{1}{2} \quad \text{for } \forall\alpha < \alpha^4(c_1) \tag{179}$$

$$\bar{\lambda}(0) < \bar{\lambda}(\frac{1}{2}) \quad \text{for } \forall\alpha < \alpha^5(c_1) \tag{180}$$

$$\bar{\lambda}''(\frac{1}{2}) < 0 \quad \text{for } \forall\alpha < \alpha^6(c_1) \tag{181}$$

Then

$$\alpha^{**}(c_1) = max\{\alpha^4(c_1), \alpha^5(c_1), \alpha^6(c_1)\} \tag{182}$$

□

G Proposition 6

In section G.2, we give three lemmas, the conclusions of which will be used in the main proof in section G.1.

G.1 Main proof of Proposition 6

Define the following normed linear spaces: $\Lambda_{S1} = \{\lambda_1^S(\delta) : \lambda_1^S(\delta) \in C^1[0,1]; \lambda_1^S(\delta) \geq 0 \text{ and } \lambda_1^{S'}(\delta) \leq 0, \forall \delta \in [0,1]\}$, $\Phi_{S1} = \{\phi_1^S(\delta) : \phi_1^S(\delta) \in C^1[0,1]; 0 \leq \phi_1^S(\delta) \leq 1 \text{ and } \phi_1^{S'}(\delta) \geq 0, \forall \delta \in [0,1]; \int_0^1 \phi_1^S(\delta)d\delta = \frac{1}{2}\}$, all with the norm $\|f\| = \underset{0\leq\delta\leq 1}{max}|f(\delta)|$. We can further transform the original social welfare problem to a new one with two control variables $\lambda_1^S(\delta) \in \Lambda_{S1}$ and $\phi_1^S(\delta) \in \Phi_{S1}$, and transfer the original equilibrium constraint as follows:

New Problem:

$$[P] \qquad \underset{\lambda_1^S(\delta)\in\Lambda_{S1},\phi_1^S(\delta)\in\Phi_{S1}}{max} \quad W = \int_0^1 (\delta - 2c_1\lambda_1^{S^2}(\delta))\phi_1^S(\delta)d\delta$$

$s.t.$

$$\begin{aligned}
&H(\lambda_1^S(\delta), \phi_1^S(\delta)) \\
&= 2\alpha\phi_1^S(\delta)\int_0^1 \lambda_1^S(\delta)\phi_1^S(\delta)d\delta + 2\phi_1^S(\delta)\lambda_1^S(\delta)\int_0^{1-\delta} \lambda_1^S(\delta')\phi_1^S(\delta')d\delta' \\
&+ 2\phi_1^S(\delta)\lambda_1^S(1-\delta)\int_0^{\delta} \lambda_1^S(\delta')\phi_1^S(\delta')d\delta' - \alpha\int_0^1 \lambda_1^S(\delta)\phi_1^S(\delta)d\delta - 2\lambda_1^S(1-\delta)\int_0^{\delta} \lambda_1^S(\delta')\phi_1^S(\delta')d\delta' \\
&\equiv 0
\end{aligned}$$

If there exists a subset $\triangle^+ \subset [\frac{1}{2}, 1]$ s.t. $\lambda_1^{S*}(\delta) > 0, \forall \delta \in \triangle^+$,[20] and with uniform distribution of δ on $[0,1]$, the measure of subset $\triangle^+$ is $\int_0^1 \mathbb{1}_{\{\delta\in\triangle^+\}}(\delta)d\delta = m^+$, we choose $\hat{\epsilon} > 0$ and $\delta_{\hat{\epsilon}^2} \in (\frac{1}{2}, 1)$ such that the Lebesgue measure of the new subset $D = \triangle^+ \cap [\frac{1}{2}, \delta_{\hat{\epsilon}^2}]$ satisfies $\mu[D] = \mu[\triangle^+ \cap [\frac{1}{2}, \delta_{\hat{\epsilon}^2}]] = \hat{\epsilon}^2 < m^+$.

Based on the $\hat{\epsilon}$ and the new subset D chosen above, we can construct a new solution

[20] $\triangle^+$ may be a union of several disjoint subintervals of $[0, 1]$.

point $\left(\lambda_1^{NS*}(\delta), \phi_1^{NS*}(\delta)\right)$ as follows:

$$\lambda_1^{NS*}(\delta) = \lambda_1^{S*}(\delta) + h_{\lambda_1^{S*}}(\delta) \tag{183}$$

where

$$h_{\lambda_1^{S*}}(\delta) = \begin{cases} -\hat{\epsilon}\lambda_1^{S*}(\delta), & \forall\delta \in D \\ 0, & \forall\delta \in [0,1]nD \end{cases}$$

and

$$\phi_1^{NS*}(\delta) = \phi_1^{S*}(\delta) + h_{\phi_1^S}(\delta) \tag{184}$$

where the incremental $h_{\phi_1^S}(\delta)$ is obtained from the following equation given the incremental

$h_{\lambda_1^{S*}}(\delta)$ chosen above:

$$
\begin{aligned}
&\frac{\partial H}{\partial \phi_1^S(\delta)} h_{\phi_1^S}(\delta) + \frac{\partial H}{\partial \lambda_1^{S*}(\delta)} h_{\lambda_1^{S*}}(\delta) \qquad (185)\\
&= \lim_{m\to 0} \left\{ \frac{H(\lambda_1^{S*}(\delta), \phi_1^S(\delta) + m h_{\phi_1^S}(\delta)) - H(\lambda_1^{S*}(\delta), \phi_1^S(\delta))}{m} \right.\\
&\left. + \frac{H(\lambda_1^{S*}(\delta) + m h_{\lambda_1^{S*}}(\delta), \phi_1^S(\delta)) - H(\lambda_1^{S*}(\delta), \phi_1^S(\delta))}{m} \right\}\\
&= 2\alpha\phi_1^S(\delta) \int_0^1 \lambda_1^{S*}(\delta) h_{\phi_1^S}(\delta) d\delta + 2\alpha h_{\phi_1^S}(\delta) \int_0^1 \lambda_1^{S*}(\delta)\phi_1^S(\delta) d\delta\\
&+ 2\phi_1^S(\delta)\lambda_1^{S*}(\delta) \int_0^{1-\delta} \lambda_1^{S*}(\delta') h_{\phi_1^S}(\delta') d\delta' + 2h_{\phi_1^S}(\delta)\lambda_1^{S*}(\delta) \int_0^{1-\delta} \lambda_1^{S*}(\delta')\phi_1^S(\delta') d\delta'\\
&+ 2\phi_1^S(\delta)\lambda_1^{S*}(1-\delta) \int_0^{\delta} \lambda_1^{S*}(\delta') h_{\phi_1^S}(\delta') d\delta' + 2h_{\phi_1^S}(\delta)\lambda_1^{S*}(1-\delta) \int_0^{\delta} \lambda_1^{S*}(\delta')\phi_1^S(\delta') d\delta'\\
&- \alpha \int_0^1 \lambda_1^{S*}(\delta) h_{\phi_1^S}(\delta) d\delta - 2\lambda_1^{S*}(1-\delta) \int_0^{\delta} \lambda_1^{S*}(\delta') h_{\phi_1^S}(\delta') d\delta'\\
&+ 2\alpha\phi_1^S(\delta) \int_0^1 h_{\lambda_1^{S*}}(\delta)\phi_1^S(\delta) d\delta + 2\phi_1^S(\delta) h_{\lambda_1^{S*}}(\delta) \int_0^{1-\delta} \lambda_1^{S*}(\delta')\phi_1^S(\delta') d\delta'\\
&+ 2\phi_1^S(\delta)\lambda_1^{S*}(\delta) \int_0^{1-\delta} h_{\lambda_1^{S*}}(\delta')\phi_1^S(\delta') d\delta' + 2\phi_1^S(\delta)\lambda_1^{S*}(1-\delta) \int_0^{\delta} h_{\lambda_1^{S*}}(\delta')\phi_1^S(\delta') d\delta'\\
&+ 2\phi_1^S(\delta) h_{\lambda_1^{S*}}(1-\delta) \int_0^{\delta} \lambda_1^{S*}(\delta')\phi_1^S(\delta') d\delta' - \alpha \int_0^1 h_{\lambda_1^{S*}}(\delta)\phi_1^S(\delta) d\delta\\
&- 2\lambda_1^{S*}(1-\delta) \int_0^{\delta} h_{\lambda_1^{S*}}(\delta')\phi_1^S(\delta') d\delta' - 2h_{\lambda_1^{S*}}(1-\delta) \int_0^{\delta} \lambda_1^{S*}(\delta')\phi_1^S(\delta') d\delta'\\
&\equiv 0 \quad \text{on } \delta \in [0,1]
\end{aligned}
$$

Then we construct another subset $B \subset [0, \frac{1}{2}]$ which is symmetric with respect to the subset D chosen above, i.e. for $\forall\delta \in B$, $1-\delta \in D$ and for $\forall\delta \in D$, $1-\delta \in B$. We will show the new solution point $\left(\lambda_1^{NS*}(\delta), \phi_1^{NS*}(\delta)\right)$ dominates the old one $\left(\lambda_1^{S*}(\delta), \phi_1^{S*}(\delta)\right)$ in the sense that the new point generates a higher value of social welfare without violating the constraint. In the proof, we need to use the conclusions of Lemma 2, Lemma 3 and Lemma 4, the proof of which will be given after the main proof.

Given the chosen $h_{\lambda_1^{S*}}(\delta)$ above and the obtained $h_{\phi_1^S}(\delta)$ from $\frac{\partial H}{\partial \phi_1^S(\delta)} h_{\phi_1^S}(\delta) + \frac{\partial H}{\partial \lambda_1^{S*}(\delta)} h_{\lambda_1^{S*}}(\delta) \equiv 0, \forall \delta \in [0,1]$ accordingly, the marginal change in the value of objective function (the social welfare) taking the chosen $\hat{\epsilon}$ to zero is:

$$\begin{aligned}
&\lim_{\hat{\epsilon}\to 0} \frac{1}{\hat{\epsilon}} \left(\frac{\partial W}{\partial \phi_1^S(\delta)} h_{\phi_1^S}(\delta) + \frac{\partial W}{\partial \lambda_1^{S*}(\delta)} h_{\lambda_1^{S*}}(\delta) \right) \\
&= \lim_{\hat{\epsilon}\to 0} \frac{1}{\hat{\epsilon}} \int_0^1 (\delta - 2c_1 \lambda_1^{S*2}(\delta)) h_{\phi_1^S}(\delta) d\delta + 4c_1 \int_D \lambda_1^{S*2}(\delta') \phi_1^S(\delta') d\delta' \\
&= \lim_{\hat{\epsilon}\to 0} \frac{1}{\hat{\epsilon}} \int_{B\cup D} (\delta - 2c_1 \lambda_1^{S*2}(\delta)) h_{\phi_1^S}(\delta) d\delta + 4c_1 \int_D \lambda_1^{S*2}(\delta') \phi_1^S(\delta') d\delta' \quad \text{(by Lemma 2)}
\end{aligned} \tag{186}$$

Also by Lemma 2 and Lemma 3:

$$\begin{aligned}
&\lim_{\hat{\epsilon}\to 0} \int_{B\cup D} (\delta - 2c_1 \lambda_1^{S*2}(\delta)) h_{\phi_1^S}(\delta) d\delta \\
&= \lim_{\hat{\epsilon}\to 0} \left(\int_B (\delta - 2c_1 \lambda_1^{S*2}(\delta)) h_{\phi_1^S}(\delta) d\delta + \int_D (\delta - 2c_1 \lambda_1^{S*2}(\delta)) h_{\phi_1^S}(\delta) d\delta \right) \\
&= \lim_{\hat{\epsilon}\to 0} \left(- \int_B (\delta - 2c_1 \lambda_1^{S*2}(\delta)) h_{\phi_1^S}(1-\delta) d\delta + \int_D (\delta - 2c_1 \lambda_1^{S*2}(\delta)) h_{\phi_1^S}(\delta) d\delta \right) \\
&= \lim_{\hat{\epsilon}\to 0} \left(- \int_D (1 - \delta - 2c_1 \lambda_1^{S*2}(1-\delta)) h_{\phi_1^S}(\delta) d\delta + \int_D (\delta - 2c_1 \lambda_1^{S*2}(\delta)) h_{\phi_1^S}(\delta) d\delta \right) \\
&= \lim_{\hat{\epsilon}\to 0} \left(\int_D (2\delta - 1 + 2c_1 \lambda_1^{S*2}(1-\delta) - 2c_1 \lambda_1^{S*2}(\delta)) h_{\phi_1^S}(\delta) d\delta \right) \\
&> 0 \quad (D \subset [\frac{1}{2}, 1] \text{ and } \lambda_1^{S*\prime}(\delta) < 0)
\end{aligned} \tag{187}$$

(186)(187) lead to:

$$\lim_{\hat{\epsilon}\to 0} \frac{1}{\hat{\epsilon}} \left(\frac{\partial W}{\partial \phi_1^S(\delta)} h_{\phi_1^S}(\delta) + \frac{\partial W}{\partial \lambda_1^{S*}(\delta)} h_{\lambda_1^{S*}}(\delta) \right) > 0 \tag{188}$$

Then by Lemma 4, any point $\left(\lambda_1^S(\delta), \phi_1^S(\delta)\right)$ with $\lambda_1^S(\delta) > 0, \forall \delta \in \triangle^+$ where $\triangle^+ \subset [\frac{1}{2}, 1]$ cannot be a local extremum. □

G.2 Three Lemmas

Lemma 2: *The incremental* $h_{\phi_1^S}(\delta)$ *satisfies*

$$h_{\phi_1^S}(\delta) = \begin{cases} O(\hat{\epsilon}), & \forall \delta \in B \cup D \\ o(\hat{\epsilon}), & \forall \delta \in [0,1]n(B \cup D) \end{cases}$$

and

$$\lim_{\hat{\epsilon}\to 0} h_{\phi_1^S}(\delta) \begin{cases} > 0, & \forall \delta \in D \\ < 0, & \forall \delta \in B \end{cases}$$

Proof:

We use guess and verify approach. We guess $h_{\phi_1^S}(\delta) = O(\hat{\epsilon}), \forall \delta \in B \cup D$, and $h_{\phi_1^S}(\delta) = o(\hat{\epsilon}), \forall \delta \in [0,1]n(B \cup D)$. Divide both sides of $\frac{\partial H}{\partial \phi_1^S(\delta)} h_{\phi_1^S}(\delta) + \frac{\partial H}{\partial \lambda_1^{S*}(\delta)} h_{\lambda_1^{S*}}(\delta) \equiv 0$ by $\hat{\epsilon}$ and take $\hat{\epsilon}$ to zero, we get:

$$\begin{aligned}
&\lim_{\hat{\epsilon}\to 0} \Bigg\{ 2\alpha\phi_1^S(\delta) \int_0^1 \lambda_1^{S*}(\delta) \frac{h_{\phi_1^S}(\delta)}{\hat{\epsilon}} d\delta + 2\alpha \frac{h_{\phi_1^S}(\delta)}{\hat{\epsilon}} \int_0^1 \lambda_1^{S*}(\delta)\phi_1^S(\delta) d\delta \\
&+ 2\phi_1^S(\delta)\lambda_1^{S*}(\delta) \int_0^{1-\delta} \lambda_1^{S*}(\delta') \frac{h_{\phi_1^S}(\delta')}{\hat{\epsilon}} d\delta' + 2\frac{h_{\phi_1^S}(\delta)}{\hat{\epsilon}} \lambda_1^{S*}(\delta) \int_0^{1-\delta} \lambda_1^{S*}(\delta')\phi_1^S(\delta') d\delta' \\
&+ 2\phi_1^S(\delta)\lambda_1^{S*}(1-\delta) \int_0^{\delta} \lambda_1^{S*}(\delta') \frac{h_{\phi_1^S}(\delta')}{\hat{\epsilon}} d\delta' + 2\frac{h_{\phi_1^S}(\delta)}{\hat{\epsilon}} \lambda_1^{S*}(1-\delta) \int_0^{\delta} \lambda_1^{S*}(\delta')\phi_1^S(\delta') d\delta' \\
&- \alpha \int_0^1 \lambda_1^{S*}(\delta) \frac{h_{\phi_1^S}(\delta)}{\hat{\epsilon}} d\delta - 2\lambda_1^{S*}(1-\delta) \int_0^{\delta} \lambda_1^{S*}(\delta') \frac{h_{\phi_1^S}(\delta')}{\hat{\epsilon}} d\delta' \\
&+ 2\alpha\phi_1^S(\delta) \int_0^1 \frac{h_{\lambda_1^{S*}}(\delta)}{\hat{\epsilon}} \phi_1^S(\delta) d\delta + 2\phi_1^S(\delta) \frac{h_{\lambda_1^{S*}}(\delta)}{\hat{\epsilon}} \int_0^{1-\delta} \lambda_1^{S*}(\delta')\phi_1^S(\delta') d\delta' \\
&+ 2\phi_1^S(\delta)\lambda_1^{S*}(\delta) \int_0^{1-\delta} \frac{h_{\lambda_1^{S*}}(\delta')}{\hat{\epsilon}} \phi_1^S(\delta') d\delta' + 2\phi_1^S(\delta)\lambda_1^{S*}(1-\delta) \int_0^{\delta} \frac{h_{\lambda_1^{S*}}(\delta')}{\hat{\epsilon}} \phi_1^S(\delta') d\delta' \\
&+ 2\phi_1^S(\delta) \frac{h_{\lambda_1^{S*}}(1-\delta)}{\hat{\epsilon}} \int_0^{\delta} \lambda_1^{S*}(\delta')\phi_1^S(\delta') d\delta' - \alpha \int_0^1 \frac{h_{\lambda_1^{S*}}(\delta')}{\hat{\epsilon}} \phi_1^S(\delta) d\delta \\
&-2\lambda_1^{S*}(1-\delta) \int_0^{\delta} \frac{h_{\lambda_1^{S*}}(\delta')}{\hat{\epsilon}} \phi_1^S(\delta') d\delta' - 2\frac{h_{\lambda_1^{S*}}(1-\delta)}{\hat{\epsilon}} \int_0^{\delta} \lambda_1^{S*}(\delta')\phi_1^S(\delta') d\delta' \Bigg\} \\
&\equiv 0 \quad \text{on } \delta \in [0,1] \qquad (189)
\end{aligned}$$

(1) For $\forall\delta \in [0,1]n(B\cup D)$, the value of left hand side (LHS) of (189) satisfies:

$$
\begin{aligned}
&LHS_1\\
&= \underset{\hat{\epsilon}\to 0}{lim}\Big\{2\alpha\phi_1^S(\delta)\int_{B\cup D}\lambda_1^{S*}(\delta)\frac{h_{\phi_1^S}(\delta)}{\hat{\epsilon}}d\delta + 2\alpha\frac{h_{\phi_1^S}(\delta)}{\hat{\epsilon}}\int_0^1\lambda_1^{S*}(\delta)\phi_1^S(\delta)d\delta\\
&+ 2\phi_1^S(\delta)\lambda_1^{S*}(\delta)\int_0^{1-\delta}\lambda_1^{S*}(\delta')\frac{h_{\phi_1^S}(\delta')}{\hat{\epsilon}}d\delta'\\
&+ 2\frac{h_{\phi_1^S}(\delta)}{\hat{\epsilon}}\lambda_1^{S*}(\delta)\int_0^{1-\delta}\lambda_1^{S*}(\delta')\phi_1^S(\delta')d\delta' + 2\phi_1^S(\delta)\lambda_1^{S*}(1-\delta)\int_0^{\delta}\lambda_1^{S*}(\delta')\frac{h_{\phi_1^S}(\delta')}{\hat{\epsilon}}d\delta'\\
&+ 2\frac{h_{\phi_1^S}(\delta)}{\hat{\epsilon}}\lambda_1^{S*}(1-\delta)\int_0^{\delta}\lambda_1^{S*}(\delta')\phi_1^S(\delta')d\delta' - \alpha\int_{B\cup D}\lambda_1^{S*}(\delta)\frac{h_{\phi_1^S}(\delta)}{\hat{\epsilon}}d\delta\\
&- 2\lambda_1^{S*}(1-\delta)\int_0^{\delta}\lambda_1^{S*}(\delta')\frac{h_{\phi_1^S}(\delta')}{\hat{\epsilon}}d\delta'\\
&- 2\alpha\phi_1^S(\delta)\int_D\lambda_1^{S*}(\delta)\phi_1^S(\delta)d\delta + 2\phi_1^S(\delta)\lambda_1^{S*}(\delta)\int_0^{1-\delta}\frac{h_{\lambda_1^{S*}}(\delta')}{\hat{\epsilon}}\phi_1^S(\delta')d\delta'\\
&+ 2\phi_1^S(\delta)\lambda_1^{S*}(1-\delta)\int_0^{\delta}\frac{h_{\lambda_1^{S*}}(\delta')}{\hat{\epsilon}}\phi_1^S(\delta')d\delta' + \alpha\int_D\lambda_1^{S*}(\delta)\phi_1^S(\delta)d\delta\\
&-2\lambda_1^{S*}(1-\delta)\int_0^{\delta}\frac{h_{\lambda_1^{S*}}(\delta')}{\hat{\epsilon}}\phi_1^S(\delta')d\delta'\Big\}\\
&= \underset{\hat{\epsilon}\to 0}{lim}\Big\{\alpha(2\phi_1^S(\delta)-1)\int_{B\cup D}\lambda_1^{S*}(\delta)\frac{h_{\phi_1^S}(\delta)}{\hat{\epsilon}}d\delta\\
&+ \underbrace{2\frac{h_{\phi_1^S}(\delta)}{\hat{\epsilon}}\left(\alpha\int_0^1\lambda_1^{S*}(\delta)\phi_1^S(\delta)d\delta + \lambda_1^{S*}(\delta)\int_0^{1-\delta}\lambda_1^{S*}(\delta')\phi_1^S(\delta')d\delta'\right)}_{(*\text{-}1)}\\
&+ \underbrace{2\frac{h_{\phi_1^S}(\delta)}{\hat{\epsilon}}\left(\lambda_1^{S*}(1-\delta)\int_0^{\delta}\lambda_1^{S*}(\delta')\phi_1^S(\delta')d\delta'\right)}_{(*\text{-}2)}\\
&+ 2\phi_1^S(\delta)\lambda_1^{S*}(\delta)\int_0^{1-\delta}\lambda_1^{S*}(\delta')\frac{h_{\phi_1^S}(\delta')}{\hat{\epsilon}}d\delta' - 2\phi_1^S(1-\delta)\lambda_1^{S*}(1-\delta)\int_0^{\delta}\lambda_1^{S*}(\delta')\frac{h_{\phi_1^S}(\delta')}{\hat{\epsilon}}d\delta'\\
&- \alpha(2\phi_1^S(\delta)-1)\int_D\lambda_1^{S*}(\delta)\phi_1^S(\delta)d\delta\\
&+ 2\phi_1^S(\delta)\lambda_1^{S*}(\delta)\int_0^{1-\delta}\frac{h_{\lambda_1^{S*}}(\delta')}{\hat{\epsilon}}\phi_1^S(\delta')d\delta' - 2\phi_1^S(1-\delta)\lambda_1^{S*}(1-\delta)\int_0^{\delta}\frac{h_{\lambda_1^{S*}}(\delta')}{\hat{\epsilon}}\phi_1^S(\delta')d\delta'\\
&\equiv 0 \quad \text{on } \delta\in[0,1]n(B\cup D)
\end{aligned} \tag{190}
$$

Denote the maximum of $\left|\lambda_1^{S*}(\delta)h_{\phi_1^S}(\delta)\right|$ over $B \cup D$ as A_1, the maximum of $\left|\lambda_1^{S*}(\delta)\phi_1^S(\delta)\right|$ over D as A_2, the maximum of $\left|h_{\lambda_1^{S*}}(\delta)\phi_1^S(\delta)\right|$ over D as A_3, then except for the $(*-1)+(*-2)$ term in equation (190), all the other terms are $o(\hat{\epsilon})$ terms:

$$\begin{aligned} &\lim_{\hat{\epsilon}\to 0}\left|\alpha(2\phi_1^S(\delta)-1)\int_{B\cup D}\lambda_1^{S*}(\delta)\frac{h_{\phi_1^S}(\delta)}{\hat{\epsilon}}d\delta\right| \\ &\leq \lim_{\hat{\epsilon}\to 0} 2\left|\alpha(2\phi_1^S(\delta)-1)\right| A_1\hat{\epsilon}^2\frac{1}{\hat{\epsilon}} = \lim_{\hat{\epsilon}\to 0} 2\left|\alpha(2\phi_1^S(\delta)-1)\right| A_1\hat{\epsilon} = 0 \end{aligned} \tag{191}$$

$$\begin{aligned} &\lim_{\hat{\epsilon}\to 0}\left|2\phi_1^S(\delta)\lambda_1^{S*}(\delta)\int_0^{1-\delta}\lambda_1^{S*}(\delta')\frac{h_{\phi_1^S}(\delta')}{\hat{\epsilon}}d\delta' - 2\phi_1^S(1-\delta)\lambda_1^{S*}(1-\delta)\int_0^{\delta}\lambda_1^{S*}(\delta')\frac{h_{\phi_1^S}(\delta')}{\hat{\epsilon}}d\delta'\right| \\ &\leq \lim_{\hat{\epsilon}\to 0}\left(\left|2\phi_1^S(\delta)\lambda_1^{S*}(\delta)\right| + \left|2\phi_1^S(1-\delta)\lambda_1^{S*}(1-\delta)\right|\right)A_1\hat{\epsilon}^2\frac{1}{\hat{\epsilon}} \\ &= \lim_{\hat{\epsilon}\to 0}\left(\left|2\phi_1^S(\delta)\lambda_1^{S*}(\delta)\right| + \left|2\phi_1^S(1-\delta)\lambda_1^{S*}(1-\delta)\right|\right)A_1\hat{\epsilon} \\ &= 0 \end{aligned} \tag{192}$$

$$\lim_{\hat{\epsilon}\to 0}\left|-\alpha(2\phi_1^S(\delta)-1)\int_D\lambda_1^{S*}(\delta)\phi_1^S(\delta)d\delta\right| \leq \lim_{\hat{\epsilon}\to 0}\left|\alpha(2\phi_1^S(\delta)-1)\right| A_2\hat{\epsilon}^2 = 0 \tag{193}$$

$$\begin{aligned} &\lim_{\hat{\epsilon}\to 0}\left|2\phi_1^S(\delta)\lambda_1^{S*}(\delta)\int_0^{1-\delta}\frac{h_{\lambda_1^{S*}}(\delta')}{\hat{\epsilon}}\phi_1^S(\delta')d\delta' - 2\phi_1^S(1-\delta)\lambda_1^{S*}(1-\delta)\int_0^{\delta}\frac{h_{\lambda_1^{S*}}(\delta')}{\hat{\epsilon}}\phi_1^S(\delta')d\delta'\right| \\ &\leq \lim_{\hat{\epsilon}\to 0}\left(\left|2\phi_1^S(\delta)\lambda_1^{S*}(\delta)\right| + \left|2\phi_1^S(1-\delta)\lambda_1^{S*}(1-\delta)\right|\right)A_3\hat{\epsilon}^2\frac{1}{\hat{\epsilon}} \\ &= \lim_{\hat{\epsilon}\to 0}\left(\left|2\phi_1^S(\delta)\lambda_1^{S*}(\delta)\right| + \left|2\phi_1^S(1-\delta)\lambda_1^{S*}(1-\delta)\right|\right)A_3\hat{\epsilon} \\ &= 0 \end{aligned} \tag{194}$$

Then to make equation (190) still apply, we conclude that

$$\lim_{\hat{\epsilon}\to 0}\frac{h_{\phi_1^S}(\delta)}{\hat{\epsilon}} = 0, \quad \forall\delta \in [0,1]n(B\cup D) \tag{195}$$

(2) For $\forall\delta \in D$, the value of the left hand side (LHS) of (189) equals to the summation of LHS value in case **(1)** (LHS_1) and another extra term with incremental $h_{\lambda_1^{S*}}(\delta)$ outside the integrals:

$$\begin{aligned}
&LHS_2 \\
&= LHS_1 + \underset{\hat{\epsilon}\to 0}{lim} 2\phi_1^S(\delta)\frac{h_{\lambda_1^{S*}}(\delta)}{\hat{\epsilon}}\int_0^\delta \lambda_1^{S*}(\delta')\phi_1^S(\delta')d\delta' \\
&= \underset{\hat{\epsilon}\to 0}{lim} 2\frac{h_{\phi_1^S}(\delta)}{\hat{\epsilon}}\bigg(\alpha\int_0^1 \lambda_1^{S*}(\delta)\phi_1^S(\delta)d\delta + \lambda_1^{S*}(\delta)\int_0^{1-\delta}\lambda_1^{S*}(\delta')\phi_1^S(\delta')d\delta' \\
&\qquad\qquad + \lambda_1^{S*}(1-\delta)\int_0^\delta \lambda_1^{S*}(\delta')\phi_1^S(\delta')d\delta'\bigg) \\
&- \underset{\hat{\epsilon}\to 0}{lim} 2\phi_1^S(\delta)\frac{\hat{\epsilon}\lambda_1^{S*}(\delta)}{\hat{\epsilon}}\int_0^{1-\delta}\lambda_1^{S*}(\delta')\phi_1^S(\delta')d\delta' + o(\hat{\epsilon}) \\
&= \underset{\hat{\epsilon}\to 0}{lim} 2\frac{h_{\phi_1^S}(\delta)}{\hat{\epsilon}}\bigg(\alpha\int_0^1 \lambda_1^{S*}(\delta)\phi_1^S(\delta)d\delta + \lambda_1^{S*}(\delta)\int_0^{1-\delta}\lambda_1^{S*}(\delta')\phi_1^S(\delta')d\delta' \\
&\qquad\qquad + \lambda_1^{S*}(1-\delta)\int_0^\delta \lambda_1^{S*}(\delta')\phi_1^S(\delta')d\delta'\bigg) \\
&- \underset{\hat{\epsilon}\to 0}{lim} 2\phi_1^S(\delta)\lambda_1^{S*}(\delta)\int_0^{1-\delta}\lambda_1^{S*}(\delta')\phi_1^S(\delta')d\delta' + o(\hat{\epsilon}) \\
&\equiv 0 \quad \text{on } \delta \in D
\end{aligned} \tag{196}$$

Then we conclude that

$$\underset{\hat{\epsilon}\to 0}{lim} h_{\phi_1^S}(\delta) = O(\hat{\epsilon}) \text{ and } \underset{\hat{\epsilon}\to 0}{lim} h_{\phi_1^S}(\delta) > 0, \quad \forall\delta \in D \tag{197}$$

(3) For $\forall\delta \in B$, the value of the left hand side (LHS) of (189) equals to the summation of LHS value in case **(1)** (LHS_1) and some extra terms with incremental $h_{\lambda_1^{S*}}(1-\delta)$ outside the integrals:

$$
\begin{aligned}
&LHS_3 \\
&= LHS_1 - \underset{\hat{\epsilon}\to 0}{lim} 2\phi_1^S(1-\delta)h_{\lambda_1^{S*}}(1-\delta)\int_0^{\delta} \lambda_1^{S*}(\delta')\phi_1^S(\delta')d\delta' \\
&= \underset{\hat{\epsilon}\to 0}{lim} 2\frac{h_{\phi_1^S}(\delta)}{\hat{\epsilon}}\left(\alpha\int_0^1 \lambda_1^{S*}(\delta)\phi_1^S(\delta)d\delta + \lambda_1^{S*}(\delta)\int_0^{1-\delta}\lambda_1^{S*}(\delta')\phi_1^S(\delta')d\delta'\right. \\
&\left. + \lambda_1^{S*}(1-\delta)\int_0^{\delta}\lambda_1^{S*}(\delta')\phi_1^S(\delta')d\delta'\right) - \underset{\hat{\epsilon}\to 0}{lim} 2\phi_1^S(1-\delta)\frac{-\hat{\epsilon}\lambda_1^{S*}(1-\delta)}{\hat{\epsilon}}\int_0^{\delta}\lambda_1^{S*}(\delta')\phi_1^S(\delta')d\delta' + o(\hat{\epsilon}) \\
&= \underset{\hat{\epsilon}\to 0}{lim} 2\frac{h_{\phi_1^S}(\delta)}{\hat{\epsilon}}\left(\alpha\int_0^1 \lambda_1^{S*}(\delta)\phi_1^S(\delta)d\delta + \lambda_1^{S*}(\delta)\int_0^{1-\delta}\lambda_1^{S*}(\delta')\phi_1^S(\delta')d\delta'\right. \\
&\left. + \lambda_1^{S*}(1-\delta)\int_0^{\delta}\lambda_1^{S*}(\delta')\phi_1^S(\delta')d\delta'\right) + \underset{\hat{\epsilon}\to 0}{lim} 2\phi_1^S(1-\delta)\lambda_1^{S*}(1-\delta)\int_0^{\delta}\lambda_1^{S*}(\delta')\phi_1^S(\delta')d\delta' + o(\hat{\epsilon}) \\
&\equiv 0 \quad \text{on } \delta\in B
\end{aligned}
\tag{198}
$$

Then we conclude that

$$
\underset{\hat{\epsilon}\to 0}{lim} h_{\phi_1^S}(\delta) = O(\hat{\epsilon}) \text{ and } \underset{\hat{\epsilon}\to 0}{lim} h_{\phi_1^S}(\delta) < 0, \quad \forall\delta\in B \tag{199}
$$

□

Lemma 3: *The incremental $h_{\phi_1^S}(\delta)$ satisfies*

$$
h_{\phi_1^S}(\delta) + h_{\phi_1^S}(1-\delta) = 0, \ \forall\delta\in[0,1] \tag{200}
$$

given any form of incremental $h_{\lambda_1^{S*}}(\delta)$.

Proof:
By $\frac{\partial H}{\partial\phi_1^S(\delta)}h_{\phi_1^S}(\delta) + \frac{\partial H}{\partial\lambda_1^{S*}(\delta)}h_{\lambda_1^{S*}}(\delta) \equiv 0, \forall\delta\in[0,1]$, we use:

$$
\frac{\partial H}{\partial\phi_1^S(\delta)}h_{\phi_1^S}(\delta) + \frac{\partial H}{\partial\lambda_1^{S*}(\delta)}h_{\lambda_1^{S*}}(\delta) + \frac{\partial H}{\partial\phi_1^S(\delta)}h_{\phi_1^S}(1-\delta) + \frac{\partial H}{\partial\lambda_1^{S*}(\delta)}h_{\lambda_1^{S*}}(1-\delta) = 0, \forall\delta\in[0,1]
$$

then we can trivally get[21]:

$$h_{\phi_1^S}(\delta) + h_{\phi_1^S}(1-\delta) = 0 \quad \forall \delta \in [0,1] \tag{201}$$

for any $h_{\lambda_1^{S*}}(\delta)$. □

Lemma 4: *Let f achieve a local extremum subject to $H(x) = \theta$ at the point x_0 and assume that f and H are continuously Fréchet differentiable in an open set containing x_0 and that x_0 is a regular point of H. Then $f'(x_0)h = 0$ for all h satisfying $H'(x_0)h = \theta$. (This lemma is from "Optimization by Vector Space Methods" by Luenberger (1973), page 242.)*

H Proposition 7

By proof of Proposition 6, the cost function $C(\lambda) = c_1\lambda^2$ only appear in conditions (186)(187). Then if condition (33) applies:

$$C'(\lambda) \begin{cases} \geq 0 & \delta = 0; \\ > 0 & \forall \delta \in (0, \lambda^{ub}]. \end{cases} \tag{202}$$

then (186) becomes:

$$\begin{aligned} & \lim_{\hat{\epsilon}\to 0} \frac{1}{\hat{\epsilon}} \left(\frac{\partial W}{\partial \phi_1^S(\delta)} h_{\phi_1^S}(\delta) + \frac{\partial W}{\partial \lambda_1^{S*}(\delta)} h_{\lambda_1^{S*}}(\delta) \right) \\ & = \lim_{\hat{\epsilon}\to 0} \frac{1}{\hat{\epsilon}} \int_0^1 (\delta - 2C(\lambda_1^{S*}(\delta))) h_{\phi_1^S}(\delta) d\delta + 2 \int_D C'(\lambda_1^{S*}(\delta'))\lambda_1^{S*}(\delta')\phi_1^S(\delta') d\delta' \\ & = \lim_{\hat{\epsilon}\to 0} \frac{1}{\hat{\epsilon}} \int_{B\cup D} (\delta - 2C(\lambda_1^{S*}(\delta))) h_{\phi_1^S}(\delta) d\delta + 2 \int_D C'(\lambda_1^{S*}(\delta'))\lambda_1^{S*}(\delta')\phi_1^S(\delta') d\delta' \\ & \text{(by 2 in Section G.2)} \end{aligned} \tag{203}$$

where the second term is still positive since $D \subset [\frac{1}{2}, 1]$ and $C'(\lambda) > 0 \ \forall \delta \in (0, \lambda^{ub}]$.

[21] The result is similar as in (97).

Also by 2 and 3 in Section G.2:

$$\begin{aligned}
&\underset{\hat{\epsilon}\to 0}{lim}\int_{B\cup D}(\delta-2C(\lambda_1^{S*}(\delta)))h_{\phi_1^S}(\delta)d\delta\\
&=\underset{\hat{\epsilon}\to 0}{lim}\left(\int_B(\delta-2C(\lambda_1^{S*}(\delta)))h_{\phi_1^S}(\delta)d\delta+\int_D(\delta-2C(\lambda_1^{S*}(\delta)))h_{\phi_1^S}(\delta)d\delta\right)\\
&=\underset{\hat{\epsilon}\to 0}{lim}\left(-\int_B(\delta-2C(\lambda_1^{S*}(\delta)))h_{\phi_1^S}(1-\delta)d\delta+\int_D(\delta-2C(\lambda_1^{S*}(\delta)))h_{\phi_1^S}(\delta)d\delta\right)\\
&=\underset{\hat{\epsilon}\to 0}{lim}\left(-\int_D(1-\delta-2C(\lambda_1^{S*}(1-\delta)))h_{\phi_1^S}(\delta)d\delta+\int_D(\delta-2C(\lambda_1^{S*}(\delta)))h_{\phi_1^S}(\delta)d\delta\right)\\
&=\underset{\hat{\epsilon}\to 0}{lim}\left(\int_D(2\delta-1+2C(\lambda_1^{S*}(1-\delta))-2C(\lambda_1^{S*}(\delta)))h_{\phi_1^S}(\delta)d\delta\right)\\
&>0\quad (C'(\lambda)\geq 0, D\subset[\frac{1}{2},1] \text{ and } \lambda_1^{S*\prime}(\delta)<0)
\end{aligned}\tag{204}$$

(203)(204) still lead to:

$$\underset{\hat{\epsilon}\to 0}{lim}\frac{1}{\hat{\epsilon}}\left(\frac{\partial W}{\partial\phi_1^S(\delta)}h_{\phi_1^S}(\delta)+\frac{\partial W}{\partial\lambda_1^{S*}(\delta)}h_{\lambda_1^{S*}}(\delta)\right)>0\tag{205}$$

Then we can still get the contradiction, then we conclude that any $\lambda_1^{S*}(\delta)$ that satisfies there exists subset $\triangle^+\subset[\frac{1}{2},1]$ where $\lambda_1^{S*}(\delta)>0, \forall\delta\in\triangle^+$ cannot be the social optimal solution.

I Solution to the social planner problem with different cost functions

I.1 Convex cost function $C(\lambda)=c_1\lambda^2$

Social Optimal Solution

$$\frac{\partial L}{\partial\lambda_1(\delta)}=\frac{\frac{\alpha}{2}-\alpha\delta+\alpha c_1\lambda_1^2(\delta)-\alpha(\alpha+\lambda_1(\delta))2c_1\lambda_1(\delta)}{(\alpha+\lambda_1(\delta))^2}=0\tag{206}$$

and

$$\frac{\partial^2 L}{\partial\lambda_1^2(\delta)}=\frac{-2c_1\alpha^3-\alpha(1-2\delta)}{(\alpha+\lambda_1(\delta))^4}<0\tag{207}$$

Then solutions is:

$$\lambda_1^{S*}(\delta) = \frac{-2c_1\alpha^2 + \sqrt{4c_1^2\alpha^4 + 4c_1\alpha^2(\frac{1}{2}-\delta)}}{2c_1\alpha} \quad \forall\delta \in [0, \frac{1}{2}), \qquad \lambda_1^{S*}(\delta) \equiv 0 \quad \forall\delta \in [\frac{1}{2}, 1] \tag{208}$$

Competitive Equilibrium Solution

$$\lambda_1^*(\delta) = \frac{\int_\delta^1 \frac{\lambda_0^*(\delta')}{\Lambda_0}(\triangle V(\delta') - \triangle V(\delta))\phi_0(\delta')d\delta'}{2c_1} \tag{209}$$

$$\lambda_0^*(\delta) = \frac{\int_0^\delta \frac{\lambda_1^*(\delta')}{\Lambda_1}(\triangle V(\delta) - \triangle V(\delta'))\phi_1(\delta')d\delta'}{2c_1} \tag{210}$$

I.2 Linear cost function $C(\lambda) = c_1\lambda$

Social Optimal Solution

$$\frac{\partial L}{\partial \lambda_1(\delta)} = \frac{\frac{\alpha}{2} - \alpha\delta + \alpha C(\lambda_1(\delta)) - \alpha(\alpha + \lambda_1(\delta))C'(\lambda_1(\delta))}{(\alpha + \lambda_1(\delta))^2} = \frac{\frac{\alpha}{2} - c_1\alpha^2 - \alpha\delta}{(\alpha + \lambda_1(\delta))^2} \tag{211}$$

Then if $c_1\alpha < \frac{1}{2}$, the solution is:

$$\lambda_1^{S^*}(\delta) = \begin{cases} \lambda^{ub} & \text{if } \delta \leq \frac{1}{2} - c_1\alpha; \\ 0 & \text{if } \delta > \frac{1}{2} - c_1\alpha. \end{cases}$$

If $c_1\alpha \geq \frac{1}{2}$,

$$\lambda_1^{S^*}(\delta) \equiv 0 \quad \forall\delta \in [0, 1] \tag{212}$$

Competitive Equilibrium Solution

For competitive equilibrium solutions, given parameters c_1, α, r, $\exists\delta^*(c_1, \alpha, r)$, s.t. $\lambda_1^*(\delta) = \lambda^{ub}$ for $\forall\delta \in [0, \delta^*(c_1, \alpha, r)]$ and $\lambda_1^*(\delta) = 0$ for $\forall\delta \in (\delta^*(c_1, \alpha, r), 1]$; by symmetry, $\lambda_0^*(\delta) = \lambda^{ub}$ for $\forall\delta \in [1 - \delta^*(c_1, \alpha, r), 1]$ and $\lambda_0^*(\delta) = 0$ for $\forall\delta \in [0, 1 - \delta^*(c_1, \alpha, r))$. For simplicity to compare with social optimal solution, we give numerical case such that $1 - \delta^*(c_1, \alpha, r) < \frac{1}{2} < \delta^*(c_1, \alpha, r)$, i.e. there exists intermediation behavior in CE equilibrium.

I.3 Social optimal solution for concave cost function $C(\lambda) = c_1\lambda^p$, $p \in (0,1)$

$$
\begin{aligned}
\frac{\partial L}{\partial \lambda_1(\delta)} &= \frac{\frac{\alpha}{2} - \alpha\delta + \alpha C(\lambda_1(\delta)) - \alpha(\alpha + \lambda_1(\delta))C'(\lambda_1(\delta))}{(\alpha + \lambda_1(\delta))^2} \\
&= \frac{\alpha(\frac{1}{2} - \delta + (1-p)c_1{\lambda_1}^p(\delta) - \alpha c_1 p {\lambda_1}^{p-1}(\delta))}{(\alpha + \lambda_1(\delta))^2}
\end{aligned} \tag{213}
$$

- Case 1: $\lambda_1^{S*}(\delta)$ that satisfies the following equation is a **stationary point**:

$$
\frac{1}{2} - \delta + (1-p)c_1{\lambda_1}^p(\delta) - \alpha c_1 p {\lambda_1}^{p-1}(\delta) = 0 \tag{214}
$$

  Since

$$
\begin{aligned}
\frac{\partial^2 L}{\partial \lambda_1^2(\delta)} &= \frac{\alpha\left(\lambda_1^{p-1}(\delta)\alpha c_1 p(p^2 - 3p + 4) + \lambda_1^{p-2}(\delta)\alpha^2 c_1 p(1-p)^2\right)}{(\alpha + \lambda_1(\delta))^3} \\
&+ \frac{\alpha\left(\lambda_1^p(\delta)c_1(1-p)(p-2) + (2\delta - 1)\right)}{(\alpha + \lambda_1(\delta))^3} \\
&= \frac{\alpha\left((\alpha c_1\lambda_1^{p-1}(\delta) - (\frac{1}{2} - \delta))p + \alpha c_1 \lambda_1^{p-1}(\delta)(1-p)^2 + (\frac{1}{2} - \delta)\frac{\alpha(1-p)^2}{\lambda_1(\delta)}\right)}{(\alpha + \lambda_1(\delta))^3} \\
&> 0 \quad \text{by (214)}
\end{aligned}
$$

  Then the stationary point is local min point.

- Case 2: Since $0 < p < 1$, then $\lambda_1(\delta) \equiv 0$ is a local max point, since $\frac{\partial L}{\partial \lambda_1(\delta)}|_{\lambda_1(\delta)=0} < 0$ $\forall \delta \in [0, \frac{1}{2})$, then the social welfare trivally $W^* = 5$ for $r = 0.05$.

- Case 3: $\lambda_1(\delta) \equiv \lambda^{ub}$ is a local max point if $\frac{1}{2} - \delta + (1-p)c_1(\lambda^{ub})^p - \alpha c_1 p(\lambda^{ub})^{p-1} > 0$ for $\forall \delta \in [0, \frac{1}{2})$.

The social optimal solution for concave cost function is either $\lambda_1^{S*}(\delta) = \lambda^{ub}$ or $\lambda_1^{S*}(\delta) = 0$ on $[0, \frac{1}{2})$ depending on parameters c_1, α, r, p. (Also need to verify expost that the generated $\triangle V^S(\delta)$ satisfies $\frac{d\triangle V(\delta)}{d\delta} > 0$ for $\forall \delta \in [0,1]$.)

Numerical Example for Case 2: $\lambda^{ub} = 0.3, c_1 = 2, \alpha = 0.75, p = 0.5$ ($\lambda_1(\delta) \equiv 0$ is a local max point but $\lambda_1(\delta) \equiv \lambda^{ub}$ is not local max point)

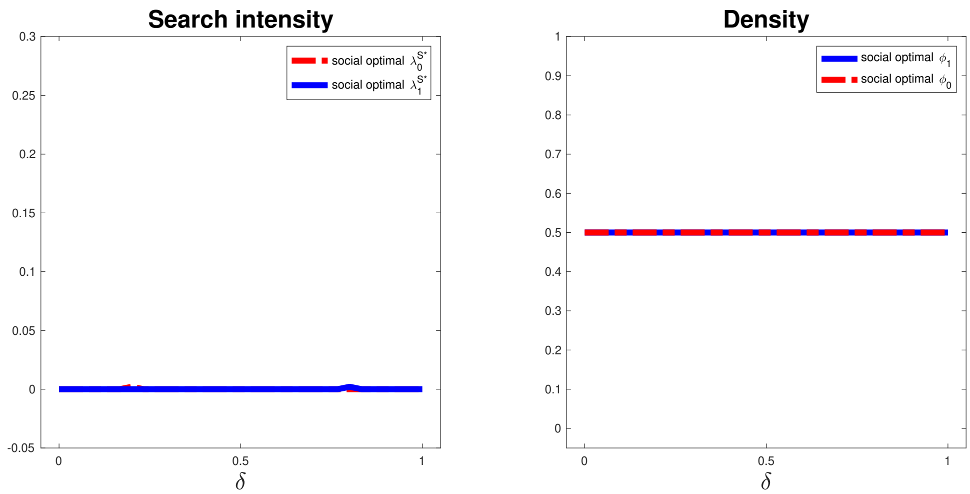

Figure 15: Case 2: Social optimal search intensities and densities for concave cost function $C(\lambda) = c_1\lambda^p$ ($\lambda^{ub} = 0.3, c_1 = 2, \alpha = 0.75, p = 0.5$)

Numerical Example for Case 3: $\lambda^{ub} = 1, c_1 = 1, \alpha = 0.05, p = 0.5$ (Both $\lambda_1(\delta) \equiv 0$ and $\lambda_1(\delta) \equiv \lambda^{ub}$ are local max points, but the marginal loss from deviating from $\lambda_1(\delta) \equiv \lambda^{ub}$ is large in this case)

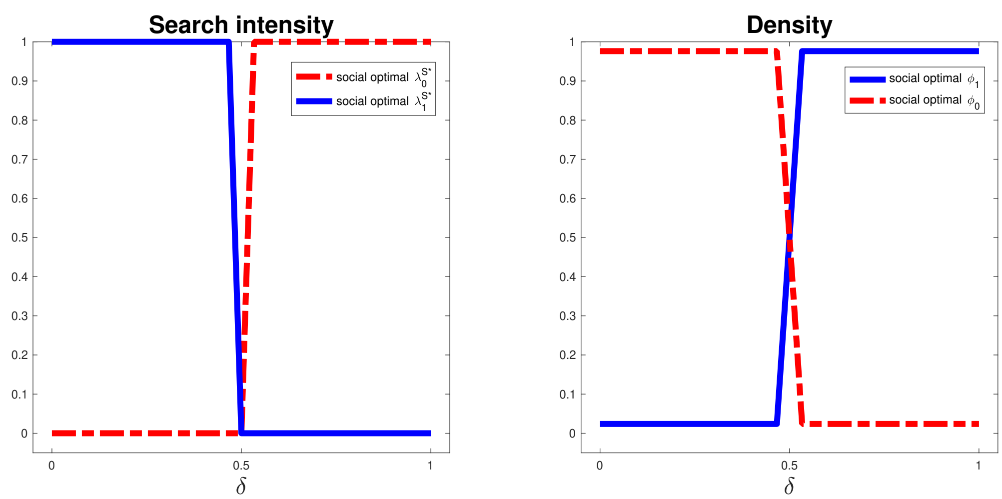

Figure 16: Case 3: Social optimal search intensities and densities for concave cost function $C(\lambda) = c_1\lambda^p$ ($\lambda^{ub} = 1, c_1 = 1, \alpha = 0.05, p = 0.5$)

I.4 Social optimal solution for $C(\lambda) = c_1\lambda^2 + c_2\lambda$ $(c_1 < 0, c_2 > 0)$

Finally, we give a numerical example for $C(\lambda) = c_1\lambda^2 + c_2\lambda$ ($c_1 < 0, c_2 > 0$ to double check the sufficient condition for $\lambda_1^{S*}(\delta) \equiv 0$ on $[\frac{1}{2}, 1]$.

$$C'(\lambda) \geq 0 \quad \forall \lambda \in [0, \lambda^{ub}] \tag{215}$$

$\Longrightarrow$

$$c_2 > -2c_1\lambda^{ub} \tag{216}$$

- Case 1: The analytical stationary point satisfies:

$$\frac{\partial L}{\partial \lambda_1(\delta)} = \frac{\frac{\alpha}{2} - \alpha\delta - \alpha^2 c_2 - 2\alpha^2 c_1 \lambda_1(\delta) - \alpha c_1 \lambda_1^2(\delta)}{(\alpha + \lambda_1(\delta))^2} = 0 \tag{217}$$

$\Longrightarrow$

$$\lambda_1{}^*(\delta) = \frac{-2\alpha c_1 + \sqrt{4\alpha^2 c_1^2 - 4c_1(\alpha c_2 + \delta - \frac{1}{2})}}{2c_1}, \quad \forall \delta \in [0, \frac{1}{2}] \quad \text{(require } \alpha^2 c_1 - \alpha c_2 + \frac{1}{2} \leq 0\text{)} \tag{218}$$

and

$$\frac{\partial^2 L}{\partial \lambda_1^2(\delta)} = \frac{2\alpha^2 c_2 - 2\alpha^3 c_1 + \alpha(2\delta - 1)}{(\alpha + \lambda_1(\delta))^4} \geq \frac{\alpha + \alpha(2\delta - 1)}{(\alpha + \lambda_1(\delta))^4} \geq 0 \quad \text{(by } \alpha^2 c_1 - \alpha c_2 + \frac{1}{2} \leq 0\text{)} \tag{219}$$

so the stationary point is a local min point.

- Case 2: If $\alpha c_2 \geq \frac{1}{2}$, then $\lambda_1{}^*(\delta) \equiv 0 \ \forall \delta \in [0, \frac{1}{2}]$ is local maximum point.

- Case 3: If $\alpha c_2 + 2\alpha c_1 \lambda^{ub} + c_1(\lambda^{ub})^2 \leq 0$, then $\lambda_1{}^*(\delta) \equiv \lambda^{ub} \ \forall \delta \in [0, \frac{1}{2}]$ is local maximum point.

Numerical Example for Case 2: $\lambda^{ub} = 2, c_1 = -0.5, c_2 = 10, \alpha = 0.5$.
Numerical Example for Case 3: $\lambda^{ub} = 1.5, c_1 = -0.5, c_2 = 2, \alpha = 0.05$.

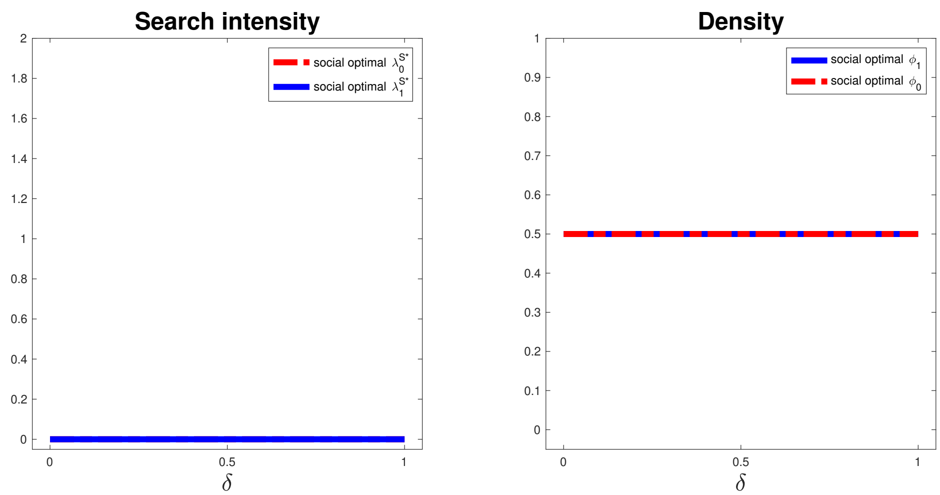

Figure 17: Case 2: Social optimal search intensities and densities for convex cost function $C(\lambda) = c_1\lambda^2 + c_2\lambda$ ($\lambda^{ub} = 2, c_1 = -0.5, c_2 = 10, \alpha = 0.5$)

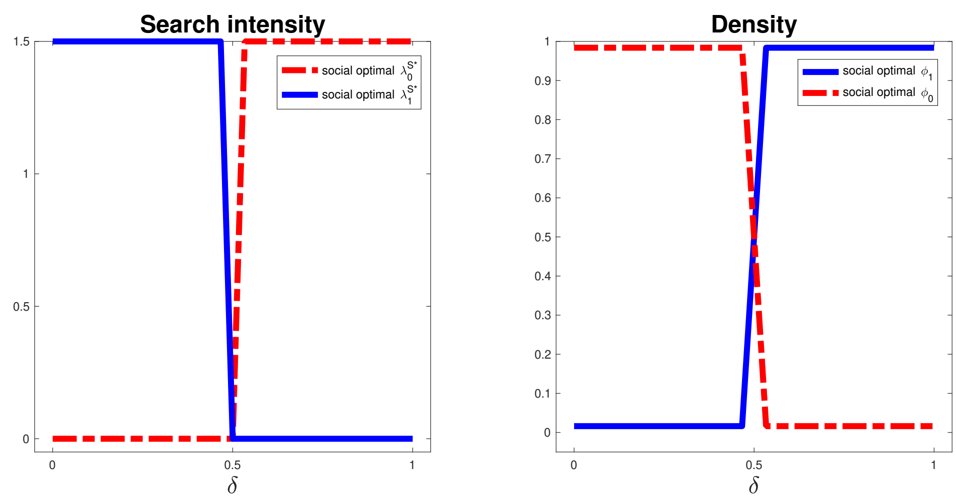

Figure 18: Case 3: Social optimal search intensities and densities for convex cost function $C(\lambda) = c_1\lambda^2 + c_2\lambda$ ($\lambda^{ub} = 1.5, c_1 = -0.5, c_2 = 2, \alpha = 0.05$)

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